# A STUDY OF THE COMPACTED SPIN COEFFICIENT FORMALISM IN GENERAL RELATIVITY

*THESIS SUBMITTED FOR THE AWARD OF THE DEGREE OF*

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By

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# CERTIFICATE

This is to certify that the contents of this thesis entitled **'A STUDY OF COMPACTED SPIN COEFFICIENT FORMALISM IN GENERAL RELATIVITY'** is the original research work of Mrs. Nikhat Ahsan (nee Nikhat Parveen Malik) carried out under my supervision.

I further certify that the work has not been submitted either partly or fully to any other University or Institution for the award of any degree.

Forwarded

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# PREFACE

The standard way of treating problems in general theory of relativity is to consider Einstein field equations in a local coordinate basis adapted to the problems with which one is working. In recent times it has proved advantageous to choose a suitable basis of four linearly independent vectors, to project the relevant quantities on to the chosen basis and consider the equations satisfied by them. This is *tetrad formalism* - and in general relativity certain types of calculations are simplified if one deals with a tetrad formalism. An important example is the Newman-Penrose formalism.

On the other hand, certain physical problems in general relativity are often conveniently described by using a formalism adopted to the geometry of the particular situation. One such formalism is the extension of Newman-Penrose formalism given by Geroch, Held and Penrose. This formalism is now known as GHP formalism or compacted spin coefficient formalism. This formalism is clearly more concise and efficient than the widely known NP formalism. The present thesis entitled **"A study of the compacted spin coefficient formalism in general relativity"** is devoted to the study of GHP formalism and its applications in general theory of relativity. It consists of five chapters and an appendix.

Chapter I deals with a study of GHP formalism. The salient features of this formalism are mentioned and a complete set of field equations, commutator relations and Bianchi identities has been given (as far as I know, this set is not available in litrature).

In order to obtain the information about the structure of the gravitational field, a study of the null congruences has been made in Chpater II. The compacted spin coefficient formalism has been used for this study. This investigation is an important activity as it brings out the geometrical meaning of the scalars characterizing the gravitational field. The contents of this Chapter have already been appeared in *Proc. Indian Acad. Sci 85 A (1977) 546-551.*

Non null electromagnetic fields via the GHP formalism is the subject of study of Chapter III. The Maxwell's equations for an arbitrary type electromagnetic field as well non null and null electromagnetic fields have been translated into the language of GHP formalism. The propagation equations for shear, expansion and twist of the null congruences have been obtained and a coupling between twist and expansion of the congruences has been established. The behaviour of the modified Lie derivative operator on the electromagnetic field bivector, Ricci tensor and metric tensor has been investigated.

The non-local part of the gravitational field in general relativity is described by the 10 components of the Weyl conformal curvature tensor. For this field, Lanczos found a potential $L_{abc}$ - now known as Lanczos potential. Chapter IV is devoted to the study of this tensor. The method of general observers has been considered and the kinematical quantities such as expansion, shear and twist etc., and the equations satisfied by them have been written in terms of the NP formalism. The tensorial versions of the earlier results of Novello and Velloso about the Lanczos potential for the perfect fluid space-times have been written in terms of the spin coefficient. A structural link between the spin coefficients and the Lanczos scalars has been established and a potential for the Gödel solution is obtained. The Weyl-Lanczos equations are translated into GHP formalism and a potential for a Petrov type D space-time is found. These results are then applied to a Kerr black hole. The contents of this Chapter has been accepted for presentation at the *21st meeting of the Indian Association for General Relativity and Gravitation, to be held at Nagpur(India), Jan. 2001.*

Once a tetrad frame $\{l^a, n^a, m^a, \bar{m}^a\}$ is chosen we can subject the frame to a Lorentz transformation at some point and extend it contineously through all of space-time. Corresponding to six parameters of the group of Lorentz transformation, we have six degrees of freedom to rotate a chosen tetrad. The transformation laws are (a) null rotation about $l^a$ (b) null rotation which leaves the direction of $l^a$ and $n^a$ unchanged, but rotate $m^a$ (and $m^a$) in $m^a - \bar{m}^a$ plane (c) null rotation about $n^a$ (d) reflection in $l^a - n^a$ plane (e) reflection in $m^a - \bar{m}^a$ plane (f) improper complex Lorentz transformation. The effects of these transformation laws on the scalars (spin coefficients, GHP derivative operators, the components of the Weyl and Ricci tensors), used to describe the gravitational field, have been examined in Chapter V. Some of the applications of these transformation laws have also been mentioned here. The contents of this Chapter have been presented at the *Conference on Recent Devalopments in Relativity and Allied Topics held at Aligarh, August*

*1999.*

In different parts of the thesis some of the results need more explaination. Such explainations and the equations required to derive these results have been mentioned in the Appendix alongwith some other related results. For example, GHP field equations, commutator relations, Bianchi identities, GHP versions of the Weyl-Lanczos equations and Lanczos differential gauge conditions have been written for different types of Petrov classification of the gravitational fields. A comparison between the electromagnetic and gravitaional theories is also given here.

The thesis ends up with a list of references which by no means is exhaustive on the subject. Only the work referred to in the thesis has been included in the list.

Mathematical relations obtained in the thesis have been numbered serially in each Chapter and so are the Theorems. Thus equation (8) refers to equation (8) in the current Chapter. If equation (8) of Chapter I is used in any subsequent Chapter it will be represented by equation (8 - I).

# CHAPTER I

# Geroch - Held - Penrose Formalism

## 1. Introduction

In the application of the tetrad formalism, the choice of the tetrad basis depends upon the symmetries of the space time with which we are working, and to some extent is the part of the problem. Such tetrad formalisms are often used in general relativity to simplify many calculations. The important example is the Newman-Penrose formalism (in short, NP formalism) [53]. This is a tetrad formalism with special choice of null basis consisting of a pair of real null vectors $l^a$ and $n^a$ and a pair of complex conjugate null vectors $m^a$ and $\bar{m}^a$. These vectors satisfy the orthogonality conditions

$$l_a m^a = l_a \bar{m}^a = n_a m^a = n_a \bar{m}^a \tag{1}$$

besides the requirement

$$l_a l^a = n_a n^a = m_a m^a = \bar{m}_a \bar{m}^a \tag{2}$$

that the vectors be null. The basis vectors also satisfy the normalization condition

$$l_a n^a = 1 \ , \ m_a \bar{m}^a = -1 \tag{3}$$

Since the chosen tetrad is null, it is not surprising that this formalism has an alternative and more general definitions in terms of spinors [55].

Although such formalisms are useful in general, they have particular advantage if the basis vectors (or, spinors) are not completely arbitrary but are related to the geometry or physics in some natural way. For example, if we are studying the geometry of a null 2- surface $S$ we can then choose the tetrad so that $l^a$ and $n^a$ point along the outgoing and ingoing normal to $S$, and the real and imaginary parts of $m^a$ and $\bar{m}^a$ are tangent to $S$. The remaining gauge freedom in the choice of the tetrad is the two dimensional subgroup of the Lorentz group representing a boost in the direction normal to $S$ and a rotation in the direction tangent to $S$. In terms of the spinors, we choose the flagpoles of $o^A$ and $\iota^A$ to point along the directions of the null normals and the remaining gauge freedom (which preserves the normalization $o_A\iota^A = 1$) is

$$o^A \longrightarrow \lambda o^A \ , \ \iota^A \longrightarrow \lambda^{-1}\iota^A \tag{4}$$

where $\lambda$ is an arbitrary (nowhere vanishing) complex scalar field. The pair of spinor fields $o^A$ and $\iota^A$ is called a *dyad* or *spin-frame.* Other important physical situations in which the tetrad is defined upto spin and boost transformations include certain radiation problems and the case of Petrov type D space times.

Under the transformation (4) some of the spin coefficients (Ricci rotation coefficients with respect to a basis) are simply rescaled, while the other transform in a way which include the derivative of $\lambda$. It turns out that the spin coefficients can be combined with the differential operators to produce new differential operators of proper spin and boost. If we deal with the quantities that simply rescale under spin and boost transformations, we then have Geroch-Held-Penrose formalism (or, in short, GHP formalism) or, compacted spin coefficient formalism [30].

The present Chapter is devoted to the study of this formalism. The salient features of the formalism are discussed in section 2; while in section 3, a complete set of field equations, commutator relations and Bianchi identities has been given ( a detailed account of this formalism and its applications alongwith the Newman-Penrose formalism have recently been given by Ahsan

[1]).

## 2. Space-time Calculus

The most general transformation, in terms of spinor notation, preserving the two preferred null directions and the dyad normalization $o_A\iota^A$ is given by equation (4). The corresponding two parameter subgroup of the Lorentz group (boost and spatial rotation) affects the complex null tetrad as follows:

$$l^a \longrightarrow rl^a \ , \ n^a \longrightarrow r^{-1}n^a \ \ \text{(boost)} \tag{5}$$

$$m^a \longrightarrow e^{i\theta}m^a \ , \ \ \text{(spatial rotation)} \tag{6}$$

where the complex vector $m^a$ is defined by $m^a = \frac{1}{\sqrt{2}}(X^a + iY^a)$, $X^a$ and $Y^a$ are the unit space-like vectors orthogonal to each of $l^a, n^a$ and to each other, and $r$ and $\theta$ are related through $\lambda^2 = re^{i\theta}$. In terms of the tetrad, the transformation (4) takes the form

$$l^a \longrightarrow \lambda\bar{\lambda}l^a \ , \ n^a \longrightarrow \lambda^{-1}\bar{\lambda}^{-1}n^a \ , \ m^a \longrightarrow \lambda\bar{\lambda}^{-1}m^a \ , \ \bar{m}^a \longrightarrow \lambda^{-1}\bar{\lambda}\bar{m}^a \tag{7}$$

The GHP formalism deals with the scalars associated with a tetrad $\{l^a, n^a, m^a, \bar{m}^a\}$ / dyad $(o^A, \iota^A)$ where the scalars undergo transformation

$$\eta \longrightarrow \lambda^p\bar{\lambda}^q\eta \tag{8}$$

whenever the tetrad/dyad is changed according to equations (5) and (6) or equations (7)/(4). Such a scalar is called a *spin* and *boost weighted scalar of type* $\{p, q\}$. The *spin weight* is $\frac{1}{2}(p-q)$ and the *boost weight* is $\frac{1}{2}(p+q)$. It may be noted that $o^A$ and $\iota^A$ may themselves be regarded as spinors of type $\{1, 0\}$ and $\{-1, 0\}$, respectively, and $l^a, n^a, m^a, \bar{m}^a$ as vectors of types $\{1, 1\}$, $\{-1, -1\}$, $\{1, -1\}$, $\{-1, 1\}$, respectively.

It is known that any dyad defines a unique null tetrad $Z^a_\mu = \{l^a, n^a, m^a, \bar{m}^a\}$ at each point and, conversely, that any null tetrad defines a dyad uniquely up to sign. The relationship is as follows:

$$l^a = o^A\bar{o}^{A'}\ ,\ n^a = \iota^A\bar{\iota}^{A'}\ ,\ m^a = o^A\bar{\iota}^{A'}\ ,\ \bar{m}^a = \iota^A\bar{o}^{A'} \tag{9}$$

The twelve spin coefficients (complex functions) are as follows:

$$\kappa = o^A\bar{o}^{A'}o^B\nabla_{AA'}o_B = m^bl^a\nabla_al_b$$

$$\sigma = o^A\bar{\iota}^{A'}o^B\nabla_{AA'}o_B = m^bm^a\nabla_al_b$$

$$\rho = \iota^A\bar{o}^{A'}o^B\nabla_{AA'}o_B = m^b\bar{m}^a\nabla_al_b$$

$$\tau = \iota^A\bar{\iota}^{A'}o^B\nabla_{AA'}o_B = m^bn^a\nabla_al_b \tag{10 a}$$

$$\kappa' = -\iota^A\bar{\iota}^{A'}\iota^B\nabla_{AA'}\iota_B = \bar{m}^bn^a\nabla_an_b$$

$$\sigma' = -\iota^A\bar{o}^{A'}\iota^B\nabla_{AA'}\iota_B = \bar{m}^b\bar{m}^a\nabla_an_b$$

$$\rho' = -o^A\bar{\iota}^{A'}\iota^B\nabla_{AA'}\iota_B = \bar{m}^bm^a\nabla_an_b$$

$$\tau' = -o^A\bar{o}^{A'}\iota^B\nabla_{AA'}\iota_B = \bar{m}^bl^a\nabla_an_b \tag{10 b}$$

$$\beta = o^A\bar{\iota}^{A'}\iota^B\nabla_{AA'}o_B = \frac{1}{2}(n^bm^a\nabla_al_b - \bar{m}^bm^a\nabla_am_b)$$

$$\beta' = -\iota^A\bar{o}^{A'}o^B\nabla_{AA'}\iota_B = \frac{1}{2}(l^b\bar{m}^a\nabla_an_b - m^b\bar{m}^a\nabla_a\bar{m}_b)$$

$$\epsilon = o^A\bar{o}^{A'}\iota^B\nabla_{AA'}o_B = \frac{1}{2}(n^bl^a\nabla_al_b - \bar{m}^bl^a\nabla_am_b)$$

$$\epsilon' = -\iota^A\bar{\iota}^{A'}o^B\nabla_{AA'}\iota_B = \frac{1}{2}(l^bn^a\nabla_an_b - m^bn^a\nabla_a\bar{m}_b) \tag{10 c}$$

We recall the definition of the tetrad (dyad) components of Weyl tensor $C_{abcd}$ (Weyl spinor $\Psi_{ABCD}$) and the trace-free Ricci tensor $R_{ab}$ (Ricci spinor $\Phi_{ABC'D'}$) :

$$\Psi_0 = -C_{abcd}l^a m^b l^c m^d = o^A o^B o^C o^D \Psi_{ABCD} = \Psi'_4$$

$$\Psi_1 = -C_{abcd}l^a n^b l^c m^d = o^A o^B o^C \iota^D \Psi_{ABCD} = \Psi'_3$$

$$\Psi_2 = -\frac{1}{2}C_{abcd}(l^a n^b l^c n^d + l^a n^b m^c \bar{m}^d) = o^A o^B \iota^C \iota^D \Psi_{ABCD} = \Psi'_2 \tag{11}$$

$$\Psi_3 = -C_{abcd}l^a n^b n^c \bar{m}^d = o^A \iota^B \iota^C \iota^D \Psi_{ABCD} = \Psi'_1$$

$$\Psi_4 = -C_{abcd}n^a \bar{m}^b n^c \bar{m}^d = \iota^A \iota^B \iota^C \iota^D \Psi_{ABCD} = \Psi'_0$$

$$\Phi_{00} = -\frac{1}{2}R_{11} = o^A o^B \bar{o}^{A'} \bar{o}^{B'} \Phi_{ABA'B'} = \bar{\Phi}_{00} = \Phi'_{22}$$

$$\Phi_{01} = -\frac{1}{2}R_{13} = o^A o^B \bar{o}^{A'} \bar{\iota}^{B'} \Phi_{ABA'B'} = \bar{\Phi}_{10} = \Phi'_{21}$$

$$\Phi_{02} = -\frac{1}{2}R_{33} = o^A o^B \bar{\iota}^{A'} \bar{\iota}^{B'} \Phi_{ABA'B'} = \bar{\Phi}_{20} = \Phi'_{20}$$

$$\Phi_{10} = -\frac{1}{2}R_{14} = o^A \iota^B \bar{o}^{A'} \bar{o}^{B'} \Phi_{ABA'B'} = \bar{\Phi}_{01} = \Phi'_{12} \tag{12}$$

$$\Phi_{11} = -\frac{1}{4}(R_{12} + R_{34}) = o^A \iota^B \bar{o}^{A'} \bar{\iota}^{B'} \Phi_{ABA'B'} = \bar{\Phi}_{11} = \Phi'_{11}$$

$$\Phi_{12} = -\frac{1}{2}R_{23} = o^A \iota^B \bar{\iota}^{A'} \bar{\iota}^{B'} \Phi_{ABA'B'} = \bar{\Phi}_{21} = \Phi'_{10}$$

$$\Phi_{20} = -\frac{1}{2}R_{44} = \iota^A\iota^B\bar{o}^{A'}\bar{o}^{B'}\Phi_{ABA'B'} = \bar{\Phi}_{02} = \Phi'_{02}$$

$$\Phi_{21} = -\frac{1}{2}R_{24} = \iota^A\iota^B\bar{o}^{A'}\bar{\iota}^{B'}\Phi_{ABA'B'} = \bar{\Phi}_{12} = \Phi'_{01}$$

$$\Phi_{22} = -\frac{1}{2}R_{22} = \iota^A\iota^B\bar{\iota}^{A'}\bar{\iota}^{B'}\Phi_{ABA'B'} = \bar{\Phi}_{22} = \Phi'_{00} \tag{12}$$

The scalar curvature is defined by

$$\Lambda \;=\; \bar{\Lambda} \;=\; \Lambda' \;=\; \frac{1}{24}R \tag{13}$$

We shall also make use of the prime systematically to denote the operation of effecting the replacement

$$o^A \longrightarrow i\iota^A \;,\; \iota^A \longrightarrow io^A \;,\; \bar{o}^{A'} \longrightarrow i\bar{\iota}^{A'} \;,\; \bar{\iota}^A \longrightarrow -i\bar{o}^{A'} \tag{14}$$

so that

$$(l^a)' \;=\; n^a \;,\; (n^a)' \;=\; l^a \;,\; (m^a)' \;=\; \bar{m}^a \;,\; (\bar{m}^a)' \;=\; m^a \tag{15}$$

This preserves normalization $o_A\iota^A = 1$ and the relationship between a quantity and its complex conjugate. Since the bar and prime commute, we can write $\bar{\eta}'$ without ambiguity. Moreover, the prime operation is involutory upto sign

$$(\eta')' \;=\; (-1)^{p+q}\eta \tag{16}$$

(for all quantities explicitly defined in this Chapter, $p+q$ is in fact even and thus this sign will play no role here).

The components of the Weyl tensor, the Ricci tensor and the spin coefficients have the spin and boost of the types as indicated below (see also Figure on page 13):

$$\Psi_0 \ : \ \{4,0\} \ , \ \Psi_1 \ : \ \{2,0\} \ , \ \Psi_2 \ : \ \{0,0\} \ , \ \Psi_3 \ : \ \{-2,0\} \ , \ \Psi_4 \ : \ \{-4,0\}$$

$$\Phi_{00} \ : \ \{2,2\} \ , \ \Phi_{01} \ : \ \{2,0\} \ , \ \Phi_{02} \ : \ \{2,-2\} \ , \ \Phi_{10} \ : \ \{0,2\} \ , \ \Phi_{11} \ : \ \{2,2\}$$

$$\Phi_{22} \ : \ \{-2,-2\} \ , \ \Phi_{21} \ : \ \{-2,0\} \ , \ \Phi_{20} \ : \ \{-2,2\} \ , \ \Phi_{12} \ : \ \{0,-2\}$$

$$\Lambda \ = \ \bar{\Lambda} \ = \ \Lambda' \ = \ \frac{1}{24}R \ : \ \{0,0\} \tag{17}$$

$$\kappa \ : \ \{3,1\} \ , \ \sigma \ : \ \{3,-1\} \ , \ \rho \ : \ \{1,1\} \ , \ \tau \ : \ \{1,-1\}$$

$$\kappa' \ : \ \{-3,-1\} \ , \ \sigma' \ : \ \{-3,1\} \ , \ \rho' \ : \ \{-1,-1\} \ , \ \tau' \ : \ \{-1,1\}$$

where the spin coefficients $\kappa', \sigma'$ etc., defined in equation (10) are related to the spin coefficients defined by Newman and Penrose [53] as follows:

$$\nu \ = \ -\kappa' \ , \ \lambda \ = \ -\sigma' \ , \ \mu \ = \ -\rho' \ , \ \pi \ = \ -\tau' \ , \ \alpha \ = \ -\beta' \ , \ \gamma \ = \ -\epsilon' \tag{18}$$

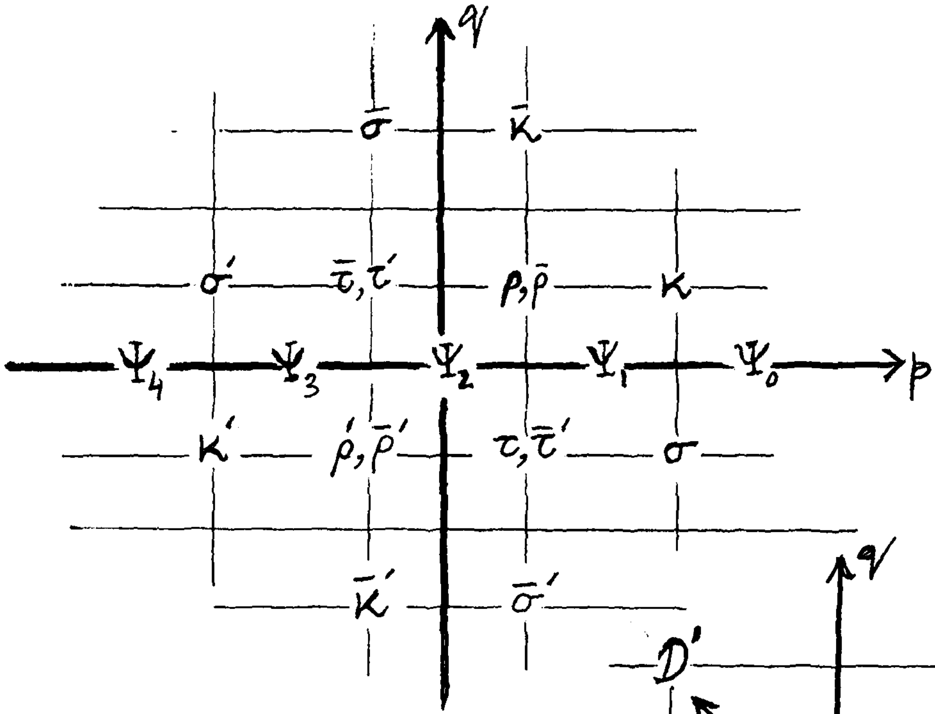
Out of the twelve spin coefficients (cf. equation (10)) only eight, given by equation (10 a) and (10 b), are of good spin and boost; and the remaining four, as defined by equation (10 c), appear in the definition of the derivatives so that the derivative may not behave badly under spin and boost transformations. For such a scalar $\eta$ of type $\{p, q\}$, the derivative operators are defined as

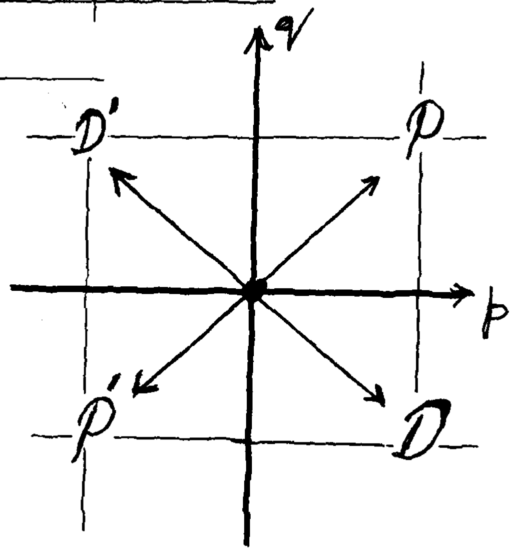
$$\mathcal{P}\eta \;=\; (D - p\epsilon - q\bar{\epsilon})\eta \;,\; \mathcal{P}'\eta \;=\; (D' + p\epsilon' + q\bar{\epsilon}')\eta$$

$$\mathcal{D}\eta \;=\; (\delta - p\beta + q\bar{\beta}')\eta \;,\; \mathcal{D}'\eta \;=\; (\delta' + p\beta' - q\bar{\beta})\eta \qquad (19)$$

where the symbols $\mathcal{P}$ and $\mathcal{D}$ are pronounced as *thorn* and *e(d)th*; $\mathcal{P}$ and $\mathcal{D}$ are the pheonetic symbols for the soft and hard *'th'*, respectively, and the types of these derivatives are

$$\mathcal{P} \,:\, \{1,1\} \;,\; \mathcal{P}' \,:\, \{-1,-1\} \;,\; \mathcal{D} \,:\, \{1,-1\} \;,\; \mathcal{D}' \,:\, \{-1,1\}$$



Alternatively, the operators may be defined in terms of the type $\{0,0\}$ operator (acting on a quantity of type $\{p,q\} = \{r+s, r-s\}$)

$$\Theta_{AA'} = \nabla_{AA'} - p\iota^B \nabla_{AA'} o^B - q\bar{\iota}^{B'} \nabla_{AA'} \bar{o}^{B'} = \nabla_a - r n^b \nabla_a l_b + s \bar{m}^b \nabla_a m_b \quad (19\ a)$$

by the equation

$$\Theta_a \ = \ l_a \mathcal{P}' \ + \ n_a \mathcal{P} \ - \ m_a \mathcal{D}' \ - \ \bar{m}_a \mathcal{D} \qquad (19\ b)$$

In equation (19 a), $s$ and $r$ are the spin and boost weights, respectively. The original definition (19) can be obtained by transvecting equation (19 b) with $l^a, n^a, m^a$ and $\bar{m}^a$.

The basic quantities with which we are concern here are the eight spin coefficients $\kappa, \sigma, \rho, \tau$; $\kappa', \sigma', \rho', \tau'$ and the four differential operators $\mathcal{P}, \mathcal{D}, \mathcal{P}', \mathcal{D}'$. There is the operation of complex conjugation and also we may consider the prime as effectively an allowable operation on the system.

The effect of the derivative operators in equation (19) is shown in the Figure on the preceeding page. With a scalar of type $\{p,q\}$ we can associate the point with coordinates $\{p,q\}$ in the plane. Each of the derivative operators in equation (19) has a characteristic effect on the type, which can be represented as a displacement in this diagram. It may be noted that the multiplication of two elements leads to a vector sum in the diagram, and if the two elements are added together then they must be represented by the same point in the diagram. Since the complex conjugate of an element of type $\{p,q\}$ is an element of type $\{q,p\}$, the operation of complex conjugation is represented by a reflection in the line $p=q$. In fact, we define

$$\bar{\mathcal{P}} \ = \ \mathcal{P} \ , \ \bar{\mathcal{P}}' \ = \ \mathcal{P}' \ , \ \bar{\mathcal{D}} \ = \ \mathcal{D}' \ , \ \bar{\mathcal{D}}' \ = \ \mathcal{D} \qquad (20)$$

then the operation of complex conjugation will satisfy

$$\overline{\mathcal{P}\eta} \ = \ \bar{\mathcal{P}}\bar{\eta} \ , \ \overline{\mathcal{D}\eta} \ = \ \bar{\mathcal{D}}\bar{\eta} \qquad (21)$$

Also, if we prime an element of type $\{p, q\}$ we get an element of type $\{-p, -q\}$ and therefore the prime operation is represented in the diagram by a reflection in the origin. The prime will commute with addition, multiplication and the complex conjugate (but not equation (16)). Moreover, we have

$$(\mathcal{P}\eta)' = \mathcal{P}'\eta' \ , \ (\mathcal{P}'\eta)' = \mathcal{P}\eta' \ , \ (\mathcal{D}\eta)' = \mathcal{D}'\eta' \ , \ (\mathcal{D}'\eta)' = \mathcal{D}\eta' \tag{22}$$

## 3. GHP Equations

As a consequence of the above considerations, the Newman-Penrose equations [53], the commutator relations [53] and the Bianchi identities get new explicit forms. They contain scalars and derivative operators of good spin and boost weights only and split into two sets of equations - one being the primed version of the other.

Since the complete set of these equations is not available in litrature (as far as I know), we shall present them here.

*GHP Field Equations*

$$\mathcal{P}\rho - \mathcal{D}\kappa = \rho^2 + \sigma\bar{\sigma} - \bar{\kappa}\tau - \tau'\kappa + \Phi_{00} \tag{23 $a$}$$

$$\mathcal{P}'\rho' - \mathcal{D}'\kappa' = \rho'^2 + \sigma'\bar{\sigma}' - \bar{\kappa}'\tau' - \tau\kappa' + \Phi_{22} \tag{23 $a'$}$$

$$\mathcal{P}\sigma - \mathcal{D}\kappa = (\rho + \bar{\rho})\sigma - (\tau + \bar{\tau}')\kappa + \Psi_0 \tag{23 $b$}$$

$$\mathcal{P}'\sigma' - \mathcal{D}'\kappa' = (\rho' + \bar{\rho}')\sigma' - (\tau' + \bar{\tau})\kappa' + \Psi_4 \tag{23 $b'$}$$

$$\mathcal{P}\tau - \mathcal{P}'\kappa = (\tau - \bar{\tau}')\rho + (\bar{\tau} - \tau')\sigma + \Psi_1 + \Phi_{01} \tag{23 $c$}$$

$$\mathcal{P}'\tau' - \mathcal{P}\kappa' = (\tau' - \bar{\tau})\rho' + (\bar{\tau}' - \tau)\sigma' + \Psi_3 + \Phi_{21} \tag{23 $c'$}$$

$$\mathcal{D}\rho - \mathcal{D}'\sigma = (\rho - \bar{\rho})\tau + (\rho' - \bar{\rho}')\kappa + \Psi_1 + \Phi_{01} \tag{23 $d$}$$

$$\mathcal{D}'\rho' - \mathcal{D}\sigma' = (\rho' - \bar{\rho}')\tau' + (\rho - \bar{\rho})\kappa' + \Psi_3 + \Phi_{21} \tag{23 $d'$}$$

$$\mathcal{D}\tau - \mathcal{P}'\sigma = -\rho'\sigma - \bar{\sigma}'\rho + \tau^2 + \kappa\bar{\kappa}' + \Phi_{02} \tag{23 $e$}$$

$$\mathcal{D}'\tau' - \mathcal{P}\sigma' = -\rho\sigma' - \bar{\sigma}\rho' + \tau'^2 + \kappa'\bar{\kappa} + \Phi_{20} \tag{23 $e'$}$$

$$\mathcal{P}'\rho - \mathcal{D}'\tau = -\rho\bar{\rho}' - \tau\bar{\tau} - \kappa\bar{\kappa}' - \Psi_2 - 2\Lambda \tag{23 $f$}$$

$$\mathcal{P}\rho' - \mathcal{D}\tau' = -\rho'\bar{\rho} - \tau'\bar{\tau}' - \kappa'\kappa - \Psi_2 - 2\Lambda \tag{23 $f'$}$$

The above list does not completely exhaust all NP field equations. The remaining equations refer to the derivatives of the spin coefficients which are spin and boost weighted quantities, and therefore can not be written like above equations. Instead they play their role as part of the commutator equations for the differential operators $\mathcal{P}, \mathcal{P}', \mathcal{D}, \mathcal{D}'$. These commutators when applied to a spin and boost weighted quantity $\eta$ of type $\{p, q\}$, are given as follows :

*GHP Commutator Relations*

$$\begin{aligned}[\mathcal{P}, \mathcal{P}']\eta = \{&(\bar{\tau} - \tau')\mathcal{D} + (\tau - \bar{\tau}')\mathcal{D}' - p(\kappa\kappa' - \tau\tau' + \Psi_2 + \Phi_{11} - \Lambda) \\ &-q(\bar{\kappa}\bar{\kappa}' - \bar{\tau}\bar{\tau}' + \Psi_2 + \Phi_{11} - \Lambda)\}\eta\end{aligned} \tag{24 $a$}$$

$$\begin{aligned}[\mathcal{P}, \mathcal{D}]\eta = \{&\bar{\rho}\mathcal{D} + \sigma\mathcal{D}' - \bar{\tau}'\mathcal{P} - \kappa\mathcal{P}' - p(\rho'\kappa - \tau'\sigma + \Psi_1) \\ &-q(\bar{\sigma}'\bar{\kappa} - \bar{\rho}\bar{\tau}' + \Phi_{01}\}\eta\end{aligned} \tag{24 $b$}$$

$$[\mathcal{D},\mathcal{D}']\eta = \{(\bar{\rho}' - \rho')\mathcal{P} + (\rho - \bar{\rho}')\mathcal{P}' - p(\rho\rho' - \sigma\sigma' + \Psi_2 - \Phi_{11} - \Lambda)$$

$$-q(\bar{\rho}\bar{\rho}' - \bar{\sigma}\bar{\sigma}' + \Psi_2 - \Phi_{11} - \Lambda)\}\eta \qquad (24\ c)$$

together with the remaining commutator relations obtained by applying prime, complex conjugation, and both to equation (24 b). Care must be taken when applying primes and bars to these equations, as $\eta', \bar{\eta}$ and $\bar{\eta}'$ have types different to that of $\eta$. Under the prime, $p$ becomes $-p$ and $q$ becomes $-q$; under bar, $p$ becomes $q$ and $q$ becomes $p$; under both bar and prime, $p$ becomes $-q$ and $q$ becomes $-p$.

*GHP Bianchi Identities (Full)*

$$\mathcal{P}\Psi_1 - \mathcal{D}'\Psi_0 - \mathcal{P}\Phi_{01} + \mathcal{D}\Phi_{00} = -\tau'\Psi_0 + 4\rho\Psi_1 - 3\kappa\Psi_2$$

$$+\bar{\tau}'\Phi_{00} - 2\bar{\rho}\Phi_{01} - 2\sigma\Phi_{10} + 2\kappa\Phi_{11} - \bar{\kappa}\Phi_{02} \qquad (25\ a)$$

$$\mathcal{P}'\Psi_3 - \mathcal{D}\Psi_4 - \mathcal{P}'\Phi_{21} + \mathcal{D}'\Phi_{22} = -\tau\Psi_4 + 4\rho'\Psi_3 - 3\kappa'\Psi_2$$

$$+\bar{\tau}\Phi_{22} - 2\bar{\rho}'\Phi_{21} - 2\sigma'\Phi_{12} + 2\kappa'\Phi_{11} - \bar{\kappa}'\Phi_{20} \qquad (25\ a')$$

$$\mathcal{P}\Psi_2 - \mathcal{D}'\Psi_1 - \mathcal{D}'\Phi_{01} + \mathcal{P}'\Phi_{00} + 2\mathcal{P}\Lambda = \sigma'\Psi_0 - 2\tau\Psi_1 + 3\rho\Psi_2$$

$$-3\kappa\Psi_3 - \bar{\rho}'\Phi_{00} - 2\bar{\tau}\Phi_{10} - 2\tau\Phi_{10} + 2\rho\Phi_{11} + \bar{\sigma}\Phi_{02} \qquad (25\ b)$$

$$\mathcal{P}'\Psi_2 - \mathcal{D}\Psi_3 - \mathcal{D}\Phi_{21} + \mathcal{P}\Phi_{22} + 2\mathcal{P}'\Lambda = \sigma\Psi_4 - 2\tau'\Psi_3 + 3\rho'\Psi_2$$

$$-3\kappa'\Psi_1 - \bar{\rho}\Phi_{22} - 2\bar{\tau}'\Phi_{21} - 2\tau'\Phi_{12} + 2\rho'\Phi_{11} + \bar{\sigma}'\Phi_{20} \qquad (25\ b')$$

$$\mathcal{P}\Psi_4 - \mathcal{D}'\Psi_3 - \mathcal{D}'\Phi_{21} + \mathcal{P}'\Phi_{20} = 3\sigma'\Psi_2 - 4\tau'\Psi_3 + \rho\Psi_4$$

$$-2\kappa'\Phi_{10} + 2\sigma'\Phi_{11} + \bar{\rho}'\Phi_{20} - 2\bar{\tau}\Phi_{21} + \bar{\sigma}\Phi_{22} \qquad (25\ c)$$

$$\mathcal{P}'\Psi_0 - \mathcal{D}\Psi_1 - \mathcal{D}\Phi_{01} + \mathcal{P}\Phi_{02} = 3\sigma\Psi_2 - 4\tau\Psi_1 + \rho'\Psi_0$$
$$-2\kappa\Phi_{12} + 2\sigma\Phi_{11} + \bar{\rho}\Phi_{02} - 2\bar{\tau}'\Phi_{01} + \bar{\sigma}'\Phi_{00} \qquad (25\ c')$$

$$\mathcal{P}\Psi_3 - \mathcal{D}'\Psi_2 - \mathcal{P}\Phi_{21} + \mathcal{D}\Phi_{20} - 2\mathcal{D}'\Lambda = 2\sigma'\Psi_1 - 3\tau'\Psi_2 + 2\rho\Psi_3$$
$$-\kappa\Psi_4 - 2\rho'\Phi_{10} + 2\tau'\Phi_{11} + \bar{\tau}'\Phi_{20} - 2\bar{\rho}\Phi_{21} + \bar{\kappa}\Phi_{22} \qquad (25\ d)$$

$$\mathcal{P}'\Psi_1 - \mathcal{D}\Psi_2 - \mathcal{P}'\Phi_{01} + \mathcal{D}'\Phi_{02} - 2\mathcal{D}\Lambda = 2\sigma\Psi_3 - 3\tau\Psi_2 + 2\rho'\Psi_1$$
$$-\kappa'\Psi_0 - 2\rho\Phi_{12} + 2\tau\Phi_{11} + \bar{\tau}\Phi_{02} - 2\bar{\rho}'\Phi_{01} + \bar{\kappa}'\Phi_{00} \qquad (25\ d')$$

*GHP Contracted Bianchi Identities*

$$\mathcal{P}\Phi_{11} - \mathcal{P}'\Phi_{00} - \mathcal{D}\Phi_{10} - \mathcal{D}'\Phi_{01} + 3\mathcal{P}\Lambda = (\rho' + \bar{\rho}')\Phi_{00} + 2(\rho + \bar{\rho})\Phi_{11}$$
$$-(\tau' + 2\bar{\tau})\Phi_{01} - (2\tau + \bar{\tau}')\Phi_{10} - \bar{\kappa}\Phi_{12} - \kappa\Phi_{21} + 2\sigma\Phi_{20} + \bar{\sigma}\Phi_{02} \qquad (26\ a)$$

$$\mathcal{P}'\Phi_{11} - \mathcal{P}\Phi_{22} - \mathcal{D}'\Phi_{12} - \mathcal{D}\Phi_{21} + 3\mathcal{P}'\Lambda = (\rho + \bar{\rho})\Phi_{22} + 2(\rho' + \bar{\rho}')\Phi_{11}$$
$$-(\tau + 2\bar{\tau}')\Phi_{21} - (2\tau' + \bar{\tau})\Phi_{12} - \bar{\kappa}'\Phi_{10} - \kappa'\Phi_{01} + 2\sigma'\Phi_{02} + \bar{\sigma}'\Phi_{20} \qquad (26\ a')$$

$$\mathcal{P}\Phi_{12} - \mathcal{P}'\Phi_{01} - \mathcal{D}\Phi_{11} - \mathcal{D}'\Phi_{02} + 3\mathcal{D}\Lambda = (\rho' + 2\bar{\rho}')\Phi_{01} + (2\rho + \bar{\rho})\Phi_{12}$$
$$-(\tau' + \bar{\tau})\Phi_{02} - 2(\tau + \bar{\tau}')\Phi_{11} - \bar{\kappa}'\Phi_{00} - \kappa\Phi_{22} + \sigma\Phi_{21} + \bar{\sigma}'\Phi_{10} \qquad (26\ b)$$

$$\mathcal{P}'\Phi_{10} - \mathcal{P}\Phi_{21} - \mathcal{D}'\Phi_{11} - \mathcal{D}\Phi_{20} + 3\mathcal{D}'\Lambda = (\rho + 2\bar{\rho})\Phi_{21} + (2\rho' + \bar{\rho}')\Phi_{10}$$
$$-(\tau + \bar{\tau}')\Phi_{20} - 2(\tau' + \bar{\tau})\Phi_{11} - \bar{\kappa}\Phi_{22} - \kappa'\Phi_{00} + \sigma'\Phi_{01} + \bar{\sigma}\Phi_{12} \qquad (26\ b')$$

The contents of the vacuum Einstein field equations can be obtained by putting $\Phi_{AB}$ and $\Lambda$ equal to zero in equations (23) and (24). The Bianchi identities (25) in this case have the following form.

*GHP Vacuum Bianchi Identities*

$$\mathcal{P}\Psi_1 - \mathcal{D}'\Psi_0 = -\tau'\Psi_0 + 4\rho\Psi_1 - 3\kappa\Psi_2 \tag{27 $a$}$$

$$\mathcal{P}'\Psi_3 - \mathcal{D}\Psi_4 = -\tau\Psi_4 + 4\rho'\Psi_3 - 3\kappa'\Psi_2 \tag{27 $a'$}$$

$$\mathcal{P}\Psi_2 - \mathcal{D}'\Psi_1 = \sigma'\Psi_0 - 2\tau\Psi_1 + 3\rho\Psi_2 - 3\kappa\Psi_3 \tag{27 $b$}$$

$$\mathcal{P}'\Psi_2 - \mathcal{D}\Psi_3 = \sigma\Psi_4 - 2\tau'\Psi_3 + 3\rho'\Psi_2 - 3\kappa'\Psi_1 \tag{27 $b'$}$$

$$\mathcal{P}\Psi_4 - \mathcal{D}'\Psi_3 = 3\sigma'\Psi_2 - 4\tau'\Psi_3 + \rho\Psi_4 \tag{27 $c$}$$

$$\mathcal{P}'\Psi_0 - \mathcal{D}\Psi_1 = 3\sigma\Psi_2 - 4\tau\Psi_1 + \rho'\Psi_0 \tag{27 $c'$}$$

$$\mathcal{P}\Psi_3 - \mathcal{D}'\Psi_2 = 2\sigma'\Psi_1 - 3\tau'\Psi_2 + 2\rho\Psi_3 - \kappa\Psi_4 \tag{27 $d$}$$

$$\mathcal{P}'\Psi_1 - \mathcal{D}\Psi_2 = 2\sigma\Psi_3 - 3\tau\Psi_2 + 2\rho'\Psi_1 - \kappa'\Psi_0 \tag{27 $d'$}$$

# CHAPTER II

## The Geometry of the Null Congruences

### 1. Introduction

There are several ways to obtain the information about the structure of the gravitational field. One of them is to study the behaviour of different test particles in the field. If the massless particles (photons, neutrinos) are considered then the trajectories corresponding to them are light rays. It is seen that from the known trajectories of massless test particles (photons) the space time metric can be recovered uniquely upto a conformal transformation. It is also known that the study of light rays and congruence of light rays play an important role in the propagation of gravitational radiation. Moreover, when the problem of algebraic classification of the electromagnetic field and Weyl tensor of a gravitational field (or, other massless fields with spin) are considered, it turns out that with every suitable tensor there is associated a set of its eigen light-like vectors. Therefore, the corresponding tensor field (electromagnetic and gravitational) generate light-like vector fields in space time. The integral curves of a light-like vector field form a congruence of null curves. By the term *congruence* we mean a family of curves (or, surfaces, etc.) of which exactly one passes through each point of a space time region under consideration.

The study of the properties of the null congruences originated in the works of Jordan, Ehlers and Sachs [37], Sachs [57] and Newman and Penrose [53] where it was shown, alongwith some other important results, that there is a set of invariants (now known as optical scalars) which determine the geometric properties of the congruence of null curves and have a simple physical meaning.

Due to the all important role of the congruence of null curves in general relativity it is worthwhile to study the geometric properties of such congruences and the present Chapter is devoted to this activity. The compacted spin coefficient formalism, discussed in Chapter I, shall be used for this study. This investigation is important as it brings out the geometric meanings of the scalars appearing in equation (10 - I). In the next section, we shall devalope a number of results that are required for our later investigations. In section 3, a number of theorems illustrating the geometric meanings of the null congruences have been proved, while a discussion of the results of this Chapter has been mentioned in section 4.

## 2. Preliminary Results

From the definition of the spin coefficients (cf. eqn. (10 - I)), we obtain

$$\nabla_a l_b = -(\mathcal{P}' - D')l_a l_b - \bar{\tau} l_a m_b - \tau l_a \bar{m}_b - (\mathcal{P} - D)n_a l_b - \bar{\kappa} n_a m_b - \kappa n_a \bar{m}_b$$
$$+(\mathcal{D}' - \delta')m_a l_b + \bar{\sigma} m_a m_b + \rho m_a \bar{m}_b + (\mathcal{D} - \delta)\bar{m}_a l_b + \bar{\rho}\bar{m}_a m_b + \sigma \bar{m}_a \bar{m}_b \quad (1)$$

$$\nabla_a n_b = (\mathcal{P}' - D')l_a n_b - \kappa' l_a m_b - \bar{\kappa} l_a \bar{m}_b - (\mathcal{P} - D)n_a n_b - \tau' n_a m_b - \bar{\tau}' n_a \bar{m}_b$$
$$-(\mathcal{D}' - \delta')m_a n_b + \sigma' m_a m_b + \bar{\rho}' m_a \bar{m}_b - (\mathcal{D} - \delta)\bar{m}_a n_b + \rho' \bar{m}_a m_b + \bar{\sigma}' \bar{m}_a \bar{m}_b \quad (2)$$

$$\nabla_a m_b = -\bar{\kappa}' l_a l_b - \tau l_a n_b + (\mathcal{P}' - D')l_a m_b - \bar{\tau}' n_a l_b - \kappa n_a n_b + (\mathcal{P} - D)n_a m_b$$
$$+\bar{\rho}' m_a l_b + \rho m_a n_b + (\mathcal{D}' - \delta')m_a m_b + \bar{\sigma}' \bar{m}_a l_b + \sigma \bar{m}_a n_b + (\mathcal{D} - \delta)\bar{m}_a m_b \quad (3)$$

$$\nabla_a \bar{m}_b = -\kappa' l_a l_b - \bar{\tau} l_a n_b + (\mathcal{P}' - D')l_a \bar{m}_b - \tau' n_a l_b - \bar{\kappa} n_a n_b + (\mathcal{P} - D)n_a \bar{m}_b$$
$$+\rho' \bar{m}_a l_b + \bar{\rho} \bar{m}_a n_b + (\mathcal{D} - \delta)\bar{m}_a \bar{m}_b + \sigma' m_a l_b + \bar{\sigma} m_a n_b + (\mathcal{D}' - \delta')m_a \bar{m}_b \quad (4)$$

where the derivatives $D,\ D',\ \delta$ and $\delta$ are defined as

$$D \;=\; l^a\nabla_a \ , \ D' \;=\; n^a\nabla_a \ , \ \delta \;=\; m^a\nabla_a \ , \ \delta' \;=\; \bar{m}^a\nabla_a \tag{5}$$

The contractions of equations (1) - (4) yield

$$\nabla_a l^a \;=\; -(\rho+\bar{\rho})-(\mathcal{P}-D) \tag{6}$$

$$\nabla_a n^a \;=\; -(\rho'-\bar{\rho}')+(\mathcal{P}'-D') \tag{7}$$

$$\nabla_a m^a \;=\; -(\bar{\tau}'+\tau)-(\mathcal{D}-\delta) \tag{8}$$

$$\nabla_a \bar{m}^a \;=\; -(\tau'+\bar{\tau})-(\mathcal{D}'-\delta') \tag{9}$$

Equations (1) - (4) also lead to the following relations

$$\begin{aligned}\nabla_{[a}l_{b]} = -2(\mathcal{P}-D)l_{[a}n_{b]} - (\bar{\tau}-\mathcal{D}'+\delta')l_{[a}m_{b]} - (\tau-\mathcal{D}+\delta)l_{[a}\bar{m}_{b]}\\ -\bar{\kappa}n_{[a}m_{b]} - \kappa n_{[a}\bar{m}_{b]} + 2i(\rho-\bar{\rho})m_{[a}\bar{m}_{b]}\end{aligned} \tag{10}$$

$$\begin{aligned}\nabla_{[a}n_{b]} = 2(\mathcal{P}-D)l_{[a}n_{b]} - \kappa' l_{[a}m_{b]} - \bar{\kappa}' l_{[a}\bar{m}_{b]} - (-\tau-\mathcal{D}'+\delta')n_{[a}m_{b]}\\ +(-\bar{\tau}'-\mathcal{D}+\delta)n_{[a}\bar{m}_{b]} - 2i(\rho'-\bar{\rho}')m_{[a}\bar{m}_{b]}\end{aligned} \tag{11}$$

$$\begin{aligned}\nabla_{[a}m_{b]} = -(\tau-\bar{\tau}')l_{[a}n_{b]} - \{2i(\mathcal{P}'-D')+\bar{\rho}'\}l_{[a}m_{b]} - \bar{\sigma}' l_{[a}\bar{m}_{b]}\\ -\{2i(\mathcal{P}-D)+\rho\}n_{[a}m_{b]} - \sigma n_{[a}\bar{m}_{b]} - (\mathcal{D}-\delta)m_{[a}\bar{m}_{b]}\end{aligned} \tag{12}$$

$$\begin{aligned}\nabla_{[a}\bar{m}_{b]} = -(\bar{\tau}-\tau')l_{[a}n_{b]} - \{2i(\mathcal{P}'-D')+\rho'\}l_{[a}\bar{m}_{b]} - \sigma' l_{[a}m_{b]}\\ -\{2i(\mathcal{P}-D)+\bar{\rho}\}n_{[a}\bar{m}_{b]} - \bar{\sigma}n_{[a}m_{b]} - (\mathcal{D}'-\delta')\bar{m}_{[a}m_{b]}\end{aligned} \tag{13}$$

When the GHP derivative operators given by equation (19 - I) act on the tetrad vectors $l^a, n^a, m^a, \bar{m}^a$, they give rise to the following set of equations.

$$(a)\ \mathcal{P}l^a = -\kappa\bar{m}^a - \bar{\kappa}m^a\ ,\ (b)\ \mathcal{P}'l^a = -\bar{\tau}m^a - \tau\bar{m}^a$$
$$(c)\ \mathcal{D}l^a = -\bar{\rho}m^a - \sigma\bar{m}^a\ ,\ (d)\ \mathcal{D}'l^a = -\bar{\sigma}m^a - \rho\bar{m}^a \quad (14)$$

$$(a)\ \mathcal{P}n^a = -\tau'm^a - \bar{\tau}'\bar{m}^a\ ,\ (b)\ \mathcal{P}'n^a = -\kappa'm^a - \bar{\kappa}'\bar{m}^a$$
$$(c)\ \mathcal{D}n^a = -\rho'm^a - \bar{\sigma}'\bar{m}^a\ ,\ (d)\ \mathcal{D}'n^a = -\bar{\rho}'\bar{m}^a - \sigma'm^a \quad (15)$$

$$(a)\ \mathcal{P}m^a = -\bar{\tau}'l^a - \kappa n^a\ ,\ (b)\ \mathcal{P}'m^a = -\bar{\kappa}'l^a - \tau n^a$$
$$(c)\ \mathcal{D}m^a = -\bar{\sigma}'l^a - \sigma n^a\ ,\ (d)\ \mathcal{D}'m^a = -\bar{\rho}'l^a - \rho n^a \quad (16)$$

$$(a)\ \mathcal{P}\bar{m}^a = -\tau'l^a - \bar{\kappa}n^a\ ,\ (b)\ \mathcal{P}'\bar{m}^a = -\kappa'l^a - \bar{\tau}n^a$$
$$(c)\ \mathcal{D}\bar{m}^a = -\sigma'l^a - \bar{\sigma}n^a\ ,\ (d)\ \mathcal{D}'\bar{m}^a = -\rho'l^a - \bar{\rho}n^a \quad (17)$$

In GHP formalism, the role of the covariant derivative operator is replaced by the operator (cf. Chapter I)

$$\Theta_a = l_a\mathcal{P}' + n_a\mathcal{P} - m_a\mathcal{D}' - \bar{m}\mathcal{D} \quad (18)$$

which leads to

$$\Theta_a l_b = (-\bar{\tau}l_a - \bar{\kappa}n_a + \bar{\sigma}m_a + \bar{\rho}\bar{m}_a)m_b + (-\tau l_a - \kappa n_a + \rho m_a + \sigma\bar{m}_a)\bar{m}_b \quad (19)$$

$$\Theta_a n_b = (-\kappa'l_a - \tau'n_a + \sigma'm_a + \rho'\bar{m}_a)m_b + (-\bar{\kappa}'l_a - \bar{\tau}'n_a + \bar{\rho}'m_a + \bar{\sigma}'\bar{m}_a)\bar{m}_b \quad (20)$$

$$\Theta_a m_b = (-\bar{\kappa}'l_a - \bar{\tau}'n_a + \bar{\rho}'m_a + \bar{\sigma}'\bar{m}_a)l_b + (-\tau l_a - \kappa n_a + \rho m_a + \sigma\bar{m}_a)n_b \quad (21)$$

$$\Theta_a \bar{m}_b = (-\kappa' l_a - \tau' n_a + \rho' \bar{m}_a + \sigma' m_a) l_b + (-\bar{\tau} l_a - \bar{\kappa} n_a + \bar{\rho} \bar{m}_a + \bar{\sigma} m_a) n_b \quad (22)$$

so that

$$\Theta_a l^a = -(\rho + \bar{\rho}) \quad (23)$$

$$\Theta_a n^a = -(\rho' + \bar{\rho}') \quad (24)$$

$$\Theta_a m^a = -(\tau + \bar{\tau}') \quad (25)$$

$$\Theta_a \bar{m}^a = -(\bar{\tau} + \tau') \quad (26)$$

From equations (14) and (18) we also have

$$\begin{aligned} l_a \Theta_b l_c = {} & l_a l_b (-\bar{\tau} m_c - \tau \bar{m}_c) + l_a n_b (-\kappa \bar{m}_c - \bar{\kappa} m_c) \\ & - l_a m_b (-\bar{\sigma} m_c - \rho \bar{m}_c) - l_a \bar{m}_b (-\bar{\rho} m_c - \sigma \bar{m}_c) \end{aligned} \quad (27)$$

$$\begin{aligned} l_b \Theta_a l_c = {} & l_b l_a (-\bar{\tau} m_c - \tau \bar{m}_c) + l_b n_a (-\kappa \bar{m}_c - \bar{\kappa} m_c) \\ & - l_b m_a (-\bar{\sigma} m_c - \rho \bar{m}_c) - l_b \bar{m}_a (-\bar{\rho} m_c - \sigma \bar{m}_c) \end{aligned} \quad (28)$$

$$\begin{aligned} l_b \Theta_c l_a = {} & l_b l_c (-\bar{\tau} m_a - \tau \bar{m}_a) + l_b n_c (-\kappa \bar{m}_a - \bar{\kappa} m_a) \\ & - l_b m_c (-\bar{\sigma} m_a - \rho \bar{m}_a) - l_b \bar{m}_c (-\bar{\rho} m_a - \sigma \bar{m}_a) \end{aligned} \quad (29)$$

$$\begin{aligned} l_c \Theta_a l_b = {} & l_c l_a (-\bar{\tau} m_b - \tau \bar{m}_b) + l_c n_a (-\kappa \bar{m}_b - \bar{\kappa} m_b) \\ & - l_c m_a (-\bar{\sigma} m_b - \rho \bar{m}_b) - l_c \bar{m}_a (-\bar{\rho} m_b - \sigma \bar{m}_b) \end{aligned} \quad (30)$$

$$l_c\Theta_b l_a = l_c l_b(-\bar{\tau}m_a - \tau\bar{m}_a) + l_c n_b(-\kappa\bar{m}_a - \bar{\kappa}m_a)$$
$$-l_c m_b(-\bar{\sigma}m_a - \rho\bar{m}_a) - l_c\bar{m}_b(-\bar{\rho}m_a - \sigma\bar{m}_a) \quad (31)$$

$$l_a\Theta_c l_b = l_a l_c(-\bar{\tau}m_b - \tau\bar{m}_b) + l_a n_c(-\kappa\bar{m}_b - \bar{\kappa}m_b)$$
$$-l_a m_c(-\bar{\sigma}m_b - \rho\bar{m}_b) - l_a\bar{m}_c(-\bar{\rho}m_b - \sigma\bar{m}_b) \quad (32)$$

so that, after simplifications, we have

$$l_{[a}\Theta_b l_{c]} = \frac{1}{3!}\{-\kappa l_a n_b\bar{m}_c - \bar{\kappa}l_a n_b m_c + \rho l_a m_b\bar{m}_c + \bar{\rho}l_a\bar{m}_b m_c$$
$$+\kappa l_a n_a\bar{m}_c + \bar{\kappa}l_b n_a m_c - \rho l_b m_a\bar{m}_c - \bar{\rho}l_b\bar{m}_a m_c$$
$$-\kappa l_b n_c\bar{m}_a - \bar{\kappa}l_b n_c m_a + \rho l_b m_c\bar{m}_a + \bar{\rho}l_b\bar{m}_c m_a$$
$$+\kappa l_c n_b\bar{m}_a + \bar{\kappa}l_c n_b m_a - \rho l_c m_b\bar{m}_a - \bar{\rho}l_c\bar{m}_b m_a$$
$$-\kappa l_c n_a\bar{m}_b - \bar{\kappa}l_c n_a m_b + \rho l_c m_a\bar{m}_b + \bar{\rho}l_c\bar{m}_a m_b$$
$$+\kappa l_a n_c\bar{m}_b + \bar{\kappa}l_a n_c m_b - \rho l_a m_c\bar{m}_b - \bar{\rho}l_a\bar{m}_c m_b\} \quad (33)$$

We also have

$$l^a\Theta_a l^b = \mathcal{P}l^b\ ,\ l^a\Theta_a n^b = \mathcal{P}n^b\ ,\ l^a\Theta_a m^b = \mathcal{P}m^b\ ,\ l^a\Theta_a\bar{m}^b = \mathcal{P}\bar{m}^b \quad (34)$$

From equations (1) and (14) it can easily be shown that

$$\nabla_a = l_a D' + n_a D - m_a\delta' - \bar{m}_a\delta \quad (35)$$

where the derivatives $D, D', \delta$ and $\delta'$ are defined by equation (5).

The duals of the skew products are

$$
\begin{aligned}
(l_{[a}n_{b]})^* &= i\ m_{[a}\bar{m}_{b]}\ ,\ (l_{[a}m_{b]})^* = -i\ l_{[a}m_{b]} \\
(l_{[a}\bar{m}_{b]})^* &= i\ l_{[a}\bar{m}_{b]}\ ,\ (n_{[a}m_{b]})^* = i\ n_{[a}m_{b]} \\
(n_{[a}\bar{m}_{b]})^* &= -i\ n_{[a}\bar{m}_{b]}\ ,\ (m_{[a}\bar{m}_{b]})^* = i\ l_{[a}n_{b]}
\end{aligned}
\tag{36}
$$

Using equation (36), the dual of equation (10) is given by

$$
\begin{aligned}
(\nabla_{[a}l_{b]})^* = -2i(\mathcal{P} - D)m_{[a}\bar{m}_{b]} + i(\bar{\tau} - \mathcal{D}' + \delta')l_{[a}m_{b]} - i(\tau - \mathcal{D} + \delta)l_{[a}\bar{m}_{b]} \\
-i\bar{\kappa}n_{[a}m_{b]} + i\kappa n_{[a}\bar{m}_{b]} - 2(\rho - \bar{\rho})l_{[a}n_{b]}
\end{aligned}
\tag{37}
$$

Thus from equations (10) and (37) we have

$$
\begin{aligned}
\nabla_{[a}l_{b]} - i(\nabla_{[a}l_{b]})^* = \{-2(\mathcal{P} - D) + 2i(\rho - \bar{\rho})\}(l_{[a}n_{b]} + m_{[a}\bar{m}_{b]}) \\
-2(\tau - \mathcal{D} + \delta)l_{[a}\bar{m}_{b]} - 2\bar{\kappa}n_{[a}m_{b]} - 2\kappa n_{[a}\bar{m}_{b]}
\end{aligned}
\tag{38}
$$

Now using the properties of the tetrad $Z^a_\mu = \{l^a, n^a, m^a, \bar{m}^a\}$ (cf. eqns.(1-3 - I)), equations (23) and (38), we get after simplification

$$
\{\nabla_{[a}l_{b]} - i(\nabla_{[a}l_{b]})^*\}l^b = -(\nabla_c l^c + 2\rho)l_a + 2\mathcal{P}l_a \tag{39}
$$

where $\mathcal{P}l_a$ is given by equation (14 a).

## 3. Geometrical Meanings of the Null Congruences

In this section, we shall prove a number of theorems illustrating the geometric properties of the spin coefficients (scalars) appearing in the GHP formalism.

**Theorem 1.** The null congruence $C(l)$ is geodesic if and only if $\kappa = 0$ and by a suitable scaling we may set $\mathcal{P} - D = 0$.

**Theorem 2.** The null congruence $C(n)$ is geodesic if and only if $\kappa' = 0$ and by a suitable scaling we may set $\mathcal{P}' - D' = 0$.

The proofs of theorems 1 and 2 follow from equations (14 a) and (15 a).

**Theorem 3.** If we choose $\mathcal{P} - D = 0$, then the tetrad $\{l^a, n^a, m^a, \bar{m}^a\}$ is parallely propagated along $C(l)$ when $\kappa = \tau' = 0$.

**Proof.** The parallel displacement along $C(l)$ requires that $l^a \Theta_a Z^b_\mu = 0$ for each $\mu$, i.e., $\mathcal{P}l^b = \mathcal{P}n^b = \mathcal{P}m^b = \mathcal{P}\bar{m}^b = 0$. From Theorem 1, $\mathcal{P}l^a = 0$ implies that $\kappa = 0$ and thus from equations (16 a) and (17 a) we have $\tau' = 0$ which proves the theorem.

Similarly we have

**Theorem 4.** If we choose $\mathcal{P}' - D' = 0$, then the tetrad $\{l^a, n^a, m^a, \bar{m}^a\}$ is parallely propagated along $C(n)$ when $\kappa' = \tau = 0$.

**Theorem 5.** The null congruence $C(l)$ is geodesic if and only if

$$l^{[b}\mathcal{P}l^{a]} = 0.$$

**Proof.** From equation (14 a) we have

$$l^b \mathcal{P} l^a = -l^b \kappa \bar{m}^a - l^b \bar{\kappa} m^a$$

$$l^a \mathcal{P} l^b = -l^a \kappa \bar{m}^b - l^a \bar{\kappa} m^b$$

which leads to

$$l^{[b} \mathcal{P} l^{a]} = -\kappa l^{[b} \bar{m}^{a]} - \bar{\kappa} l^{[b} m^{a]}$$

Since the linear dependence of the bivectors $l^{[b}m^{a]}$ and $l^{[b}\bar{m}^{a]}$ imply the vanishing of $\kappa$ and $\bar{\kappa}$, therefore, $l^{[b}\mathcal{P}l^{a]} = 0$.

In a similar manner for $C(n)$, we have

**Theorem 6.** The null congruence $C(n)$ is geodesic if and only if

$$n^{[b} \mathcal{P}' n^{a]} = 0.$$

We shall now prove the following lemma for our later investigation.

**Lemma 1.** Let $C(l)$ be a null geodesic congruence then $l_{[a}\Theta_b l_{c]} = 0$ is equivalent to $\rho = \bar{\rho}$.

**Proof.** If $C(l)$ is a null geodesic congruence then equation (33) leads to

$$l_{[a}\Theta_b l_{c]} = (\rho - \bar{\rho}) l_{[a} m_b \bar{m}_{c]} \tag{40}$$

As the vectors $l_a, n_a, m_a$ and $\bar{m}_a$ are linearly independent, the right hand side of equation (40) is non zero and thus $l_{[a}\Theta_b l_{c]} = 0$ implies that $\rho = \bar{\rho}$. Conversely, if $\rho = \bar{\rho}$ then from equation (40), $l_{[a}\Theta_b l_{c]} = 0$ which completes the proof of the lemma 1.

From Schouten [58], the condition of lemma 1 (in the language of GHP formalism) , viz. $k_{[a}\Theta_b k_{c]} = 0$ is the necessary and sufficient condition for the congruence $C(k)$ to be hypersurface forming (or hypersurface orthogonal)*[1] and thus from lemma 1, we have

**Theorem 7.** The null vector field $\mathbf{l}$ is hypersurface orthogonal if and only if $\kappa = 0$ and $\rho = \bar{\rho}$.

**Theorem 8.** The null vector field $\mathbf{n}$ is hypersurface orthogonal if and only if $\kappa' = 0$ and $\rho' = \bar{\rho}'$.

From equations (14 a) and (39) we also have

**Theorem 9.** The null congruence $C(l)$ is geodesic if and only if

$$\{\nabla_{[a}l_{b]} - i(\nabla_{[a}l_{b]})^*\}l^b = -(\nabla_c l^c + 2\rho)l_a$$

Moreover, from the properties of the tetrad $Z^a_\mu = \{l^a, n^a, m^a, \bar{m}^a\}$ and equation (37) it can easily be verified that

$$(\nabla_{[a}l_{b]})^* l^b = -\Im(\rho)l_a - \frac{i}{2}\mathcal{P}l_a \tag{41 a}$$

$$(\nabla_{[a}l_{b]})^* n^b = \Im(\rho)n_a + \frac{i}{2}\mathcal{P}'l_a - \frac{i}{2}(\mathcal{D}' - \delta')m_a - \frac{i}{2}(\mathcal{D} - \delta)\bar{m}_a \tag{41 b}$$

$$(\nabla_{[a}l_{b]})^* m^b = \frac{i}{2}(\mathcal{D} - \delta)l_a + \frac{i}{2}(\tau l_a - \kappa n_a) + i(\mathcal{P} - D)m_a \tag{41 c}$$

$$(\nabla_{[a}l_{b]})^* \bar{m}^b = -\frac{i}{2}(\mathcal{D}' - \delta')l_a - \frac{i}{2}(\bar{\tau}l_a - \bar{\kappa}n_a) - i(\mathcal{P} - D)\bar{m}_a \tag{41 d}$$

Further, if $C(l)$ is null geodesic congruence with $\mathcal{P} - D = 0$, then $l_a$ will be tangent vector corresponding to an affine parametrization and the scalars

[1]*In fact the necessary and sufficient condition for the congruence $C(k)$ to be hypersurface orthogonal, according to Schouten [58], is $k_{[a}\nabla_b k_{c]} = 0$.

$$\theta_{C(l)} = \frac{1}{2}\nabla_a l^a = -\Re(\rho) = -(\rho+\bar{\rho})$$

$$\omega_{C(l)} = -\{\frac{1}{2}\nabla_{[a}l_{b]}\,\nabla^a l^b\}^{1/2} = -\Im(\rho) = -(\rho-\bar{\rho})$$

and

$$\sigma\bar{\sigma} = |\sigma| = \frac{1}{2}\{\nabla_{[a}l_{b]}\,\nabla^a l^b - (\theta_{C(l)})^2\}^{1/2}$$

are, respectively, the *expansion*, the *twist* and the *shear* of the congruence $C(l)$.

From equations (41) we thus have

**Theorem 10.** If $C(l)$ is a null geodesic congruence with $\mathcal{P} - D = 0$ then

$$(\nabla_{[a}l_{b]})^* l^b = \omega_{C(l)} l_a = -\Im(\rho) l_a$$

$$(\nabla_{[a}l_{b]})^* n^b = \omega_{C(l)} n_a + \frac{i}{2}\{\mathcal{P}' l_a - (\mathcal{D}' - \delta')m_a + (\mathcal{D} - \delta)\bar{m}_a\}$$

$$(\nabla_{[a}l_{b]})^* m^b = \frac{i}{2}(\mathcal{D} - \delta + \tau) l_a$$

$$(\nabla_{[a}l_{b]})^* \bar{m}^b = -\frac{i}{2}(\mathcal{D}' - \delta' - \bar{\tau}) l_a$$

We conclude our investigations about the null congruences by saying that the null geodesic congruences $C(l)$ and $C(n)$ are shear-free if $\sigma = 0$ and $\sigma' = 0$, respectively.

## 4. Conclusion

A study of the null congruences within the framework of general theory of relativity, using GHP formalism, has been made. This study is important as it illustrate the geometric meanings of the spin coefficients appearing in the GHP formalism. A number of conditions for a null congruence to be geodetic have been obtained alongwith the condition of geodesicness in terms of the optical scalars. The parallel propagation of the tetrad vectors has been discussed in terms of the spin coefficients. The necessary and sufficient conditions for a congruence to be hypersurface orthogonal have also been obtained.

# CHAPTER III

## Non Null Electromagnetic Fields and the Compacted Spin Coefficient Formalism

### 1. Introduction

It is known that the Newman-Penrose formalism [53] can successfully be used in tackling a number of problems in general theory of relativity. An extension to this formalism has been given by Geroch, Held and Penrose (cf. Chapter I). This formalism is more concise and efficient than the widely knwon NP formalism, however, the GHP formalism failed to devalope its full potential to the extent to which the NP formalism has. About twenty five years ago, soon after the appearence of GHP formalism, Held ([34], [35]) proposed a simple proceedure for integration within this formalism and applied it successfully to Petrov type D vacuum metrics. The geometrical meanings of the spin coefficients appearing in this formalism have been given by Ahsan and Malik [2] (cf. Chapter II). Recently the GHP formalism has again attracted the attention of several workers and in this connection, Ludwig [47] has given an extension to this formalism by considering the quantities that transform properly under all diagonal transformation of the under lying spin-frame, i.e., not only under boost rotation but also under conformal scaling. The role of commutator relations in this extended formalism has been explored by Edgar [18]. On the other hand, Kolassis and Ludwig [41] have studied the space times which admit a two dimensional group of conformal motions (and in particular homothetic motion). The so called post Bianchi identities, which play a crucial role in search of Petrov type I solutions of Eintein field equations, have been studied by Ludwig [48] through GHP formalism. More recently, a complete proceedure for integration within this formalism has been given by Edgar, Ludwig and Vickers ([17], [22], [23], [24], [25], [49]).

Motivated by the applications of GHP formalism, in this Chapter the non null (non singular) electromagnetic fields have been studied using this formalism. In section 2, the Maxwell's equations for an arbitrary type electromagnetic field as well as non null and null electromagnetic fields have been translated in the language of GHP formalism. Various properties of the congruences associated with the non null electromagnetic field have been studied here and it is seen that the expansion and twist of the congruences can be coupled together. The behaviour of the modified Lie derivative operator on the electromagnetic field bivector, Ricci tensor and metric tensor has been investigated in section 3 and a discussion of the results of this Chapter has been mentioned in section 4.

## 2. The Maxwell's Equations and the Non Null Electromagnetic Fields

Let $M$ be a four dimensional Lorentzian manifold that admits a Lorentzian metric of signature $(- \; - \; - \; +)$. Let $Z^a_\mu = \{l^a, n^a, m^a, \bar{m}^a\}$ be the complex null tetrad satisfying the properties (1-3 - I). With these properties of the tetrad, $g_{ij}$ can be written as

$$g_{ij} = 2l_{(i}n_{j)} - 2m_{(i}\bar{m}_{j)} \tag{1}$$

In terms of the complex null tetrad $Z^a_\mu$, the electromagnetic bivector $F_{ij}$ has the following form ([13], [27])

$$F_{ij} = -2\Re\phi_1 l_{[i}n_{j]} + 2i\Im\phi_1 m_{[i}\bar{m}_{j]} + \phi_2 l_{[i}m_{j]} + \bar{\phi}_2 l_{[i}\bar{m}_{j]} - \bar{\phi}_0 n_{[i}m_{j]} - \phi_0 n_{[i}\bar{m}_{j]} \tag{2}$$

where

$$\phi_0 = 2F_{ij}l^i m^j \ , \ \phi_1 = 2F_{ij}(l^i n^j + \bar{m}^i m^j) \ , \ \phi_2 = 2F_{ij}\bar{m}^i n^j \tag{3}$$

are the complex Maxwell scalars, $\Re\phi_1$ and $\Im\phi_1$, respectively, denote the real and imaginary parts of $\phi_1$. The scalar $\phi_1$ describes the Coulomb component of the field, while the scalar $\phi_2$ arises from the electric dipole radiation of an accelerated charge. If acceleration is absent then $\phi_2 = 0$.

When a source term $J^i$ is present, the Maxwell's equations

$$\nabla_j F^{ij} = \frac{1}{2} J^i \ , \ \nabla_j F^{*\ ij} = \frac{1}{2} J^i \tag{4}$$

where $F^{ij}$ is a real bivector and $F^{*\ ij}$ is its dual, can be expressed as

$$\nabla_j N^{ij} = \frac{1}{2} J^i \tag{5}$$

where

$$N^{ij} = \frac{1}{2}(F^{ij} + F^{*\ ij}) = \phi_2 l^{[i} m^{j]} - \phi_1 (l^{[i} n^{j]} - m^{[i} \bar{m}^{j]} - \phi_0 n^{[i} \bar{m}^{j]} \tag{6}$$

and $J^i$ satisfies the conservation law

$$\nabla_i J^i = 0 \tag{7}$$

Equations (5) and (7) may be referred to as the basic equations of the electromagnetic field. From the properties of the complex null tetrad $Z^a_\mu$, we have

$$\begin{aligned} J^i &= Z^{ia} J_a = Z^i_a \eta^{ab} J_b \\ &= l^a J_2 + n^a J_1 - m^a \bar{J}_4 - \bar{m}_a J_3 \\ &= n^a I_0 + l^a I_2 - m^a \bar{I}_1 - \bar{m}_a I_1 \end{aligned} \tag{8}$$

where $I_0, I_1, \bar{I}_1, I_2$ are the source scalars. $I_0, I_2$ are real and $I_1, \bar{I}_1$ are the complex conjugates. Also from equation (8)

$$J_i J^i \;=\; J^2 \;=\; 2(I_0 I_2 - I_1 \bar{I}_1) \tag{9}$$

It is known that [44] the electromagnetic field admits two real invariants and they can be combined into a single complex invariant

$$K \;=\; \frac{1}{2}(F_{ij}F^{ij} \;+\; iF_{ij}F^{*\,ij}) \tag{10}$$

which is also equivalent to $K \;=\; N_{ij}N^{ij}$. By employing equation (10) and the Maxwell's scalars (cf. eqn. (3)), $K$ reduces to

$$K \;=\; 2(\phi_0\phi_2 \;-\; \phi_1^2) \tag{11}$$

Recall that an electromagnetic field is non null (non singular, or, algebraically general) if $K \neq 0$ and null (singular, or, algebraically special) if $K = 0$. Depending upon the vanishing of the Maxwell's scalars (3), the electromagnetic field can be classified as ([13])

$$\begin{array}{llll} \text{Type A :} & \text{non null} & (K \neq 0) & \phi_0 = \phi_2 = 0 \;,\; \phi_1 \neq 0 \\ \text{Type B :} & \text{null} & (K \;=\; 0) & \phi_0 = \phi_1 = 0 \;,\; \phi_2 \neq 0 \\ \text{Type C :} & \text{null} & (K \;=\; 0) & \phi_1 = \phi_2 = 0 \;,\; \phi_0 \neq 0 \end{array} \tag{12}$$

It may be noted that, in fact, there are just two types (types A and B). Types B and C can be transformed into each other by switching $l^a$ and $n^a$

in the null basis $\{l^a, n^a, m^a, \bar{m}^a\}$.* [1] For the sake of completeness, we have mentioned here all the three types.

Equation (12) now leads to the following forms of $F_{ij}$ ( as defined by equation (2)) for non null and null electromagnetic fields, respectively.

$$\text{Type A} \ : \ F_{ij} \ = \ -2\Re\phi_1 \ l_{[i}n_{j]} + 2i\Im\phi_1 \ m_{[i}\bar{m}_{j]} \tag{13}$$

$$\text{Type B} \ : \ F_{ij} \ = \ \phi_2 \ l_{[i}m_{j]} + \bar{\phi}_2 \ l_{[i}\bar{m}_{j]} \tag{14}$$

From equations (2), (3), (5), (6) and (19 b - I), the GHP version of the Maxwell's equations with a source term is given by

**Lemma 1.** In the presence of a source the Maxwell's equations for an electromagnetic field of arbitrary type are given by

$$\mathcal{P}\phi_1 - \mathcal{D}'\phi_0 \ = \ 2\rho\phi_1 - \tau'\phi_0 - \kappa\phi_2 + I_0 \tag{15 a}$$

$$\mathcal{D}\phi_1 - \mathcal{P}'\phi_0 \ = \ 2\tau\phi_1 - \rho'\phi_0 - \sigma\phi_2 - I_1 \tag{15 b}$$

$$\mathcal{P}\phi_2 - \mathcal{D}'\phi_1 \ = \ \rho\phi_2 - 2\tau'\phi_1 - \sigma'\phi_0 - \bar{I}_1 \tag{15 c}$$

$$\mathcal{D}\phi_2 - \mathcal{P}'\phi_1 \ = \ \tau\phi_2 - 2\rho'\phi_1 + \kappa'\phi_0 + I_2 \tag{15 d}$$

$$\mathcal{P}'I_0 + \mathcal{P}I_2 - \mathcal{D}\bar{I}_1 - \mathcal{D}'I_1 = (\rho' + \bar{\rho}')I_0 - (\bar{\tau} + \tau')I_1 - (\tau - \bar{\tau}')\bar{I}_1 + (\rho + \bar{\rho})I_2 \tag{16}$$

[1]* The electromagnetic field tensor $F_{ab}$ (in spinor language) is determined by a symmetric spinor $\Phi_{AB}$ and one can write

$$\Phi_{AB} = \alpha_A\beta_B + \alpha_B\beta_A$$

where $\alpha$ and $\beta$ are spinors. If $\alpha$ and $\beta$ are linearly independent, electromagnetic field is said to be *algebraically general* , otherwise it is *algebraically special.* According to this terminology, in fact we are studying the algebraically general electromagnetic fields in this Chapter. However, in the litrature the terms 'non null' and 'null' are commonly used for algebraically general and algebraically special electromagnetic fields, respectively.

The spin and boost types of the Maxwell's scalars $\phi_0, \phi_1$ and $\phi_2$ and the source scalars $I_0, I_1, \tilde{I}_1$ and $I_2$ are given as follows:

$$\phi_0 = -\phi_2' \quad : \{2,0\} \,,\ \phi_1 = -\phi_1' \quad : \{0,0\} \,,\ \phi_2 = -\phi_0' \quad : \{-2,0\} \quad (17\ a)$$

$$I_0 \ : \ \{1,1\} \,,\ I_1 \ : \ \{-1,1\} \,,\ \bar{I}_1 \ : \ \{1,-1\} \,,\ I_2 \ : \ \{-1,-1\} \quad (17\ b)$$

Also, the energy momentum tensor $T^{ab}$ satisfies the equation $\nabla_a T^{ab} = F_b^a J^b$ and hence $\nabla_a T^{ab} = 0$ requires that

$$F_b^a J^b = 0 \quad (18)$$

From Lemma 1, the source-free Maxwell's equations for an electromagnetic field of arbitrary type are equivalent to

$$\mathcal{P}\phi_1 - \mathcal{D}'\phi_0 = 2\rho\phi_1 - \tau'\phi_0 - \kappa\phi_2 \quad (19\ a)$$

$$\mathcal{D}\phi_1 - \mathcal{P}'\phi_0 = 2\tau\phi_1 - \rho'\phi_0 - \sigma\phi_2 \quad (19\ b)$$

$$\mathcal{P}\phi_2 - \mathcal{D}'\phi_1 = \rho\phi_2 - 2\tau'\phi_1 - \sigma'\phi_0 \quad (19\ c)$$

$$\mathcal{D}\phi_2 - \mathcal{P}'\phi_1 = \tau\phi_2 - 2\rho'\phi_1 + \kappa'\phi_0 \quad (19\ d)$$

so that from equations (12) and (19), the source-free Maxwell's equations for a non null electromagnetic field are equivalent to

$$\mathcal{P}\phi_1 = 2\rho\phi_1 \,,\ \mathcal{D}\phi_1 = 2\tau\phi_1 \,,\ \mathcal{D}'\phi_1 = 2\tau'\phi_1 \,,\ \mathcal{P}'\phi_1 = 2\rho'\phi_1 \quad (20)$$

while for null electromagnetic fields of types B and C, the source-free Maxwell's equations are, respectively, equivalent to

$$\mathcal{P}\phi_2 \;=\; \rho\phi_2 \;,\; \mathcal{D}\phi_2 \;=\; \tau\phi_2 \;,\; \kappa \;=\; \sigma \;=\; 0 \tag{21}$$

$$\mathcal{P}'\phi_0 \;=\; \rho'\phi_0 \;,\; \mathcal{D}'\phi_0 \;=\; \tau'\phi_0 \;,\; \kappa' \;=\; \sigma' \;=\; 0 \tag{22}$$

Since $\phi_2 \neq 0$ and $\phi_0 \neq 0$, the equations (21) and (22) incidently establish the Mariot-Robinson theorem [56], i.e., $\kappa = \sigma = 0$ and $\kappa' = \sigma' = 0$. It may be noted that in equation (21) the propagation vector is $l^a$ whlie in equation (22) it is $n^a$.

We shall now find the conditions that must be satisfied by the spin coefficients to admit type A (algebraically general) solutions of the Maxwell's equations.

For the existence of a solution $\phi$ of a non null electromagnetic field, the necessary and sufficient condition is that the commutators $[\mathcal{P},\mathcal{D}]$, $[\mathcal{P},\mathcal{D}']$, $[\mathcal{P}',\mathcal{D}]$, $[\mathcal{P}',\mathcal{D}']$, $[\mathcal{P},\mathcal{P}']$ and $[\mathcal{D},\mathcal{D}']$ as computed from GHP commutators (cf. eqn.(24 - I)) agree with the commutators obtained from GHP field equations (23 - I). The agreement between the commutators exists if and only if the following equations are satisfied.

$$\mathcal{P}'\kappa - \mathcal{D}'\sigma = (2\tau' - \bar{\tau})\sigma - \bar{\rho}'\kappa - 2\Psi_1 \tag{23 a}$$

$$\mathcal{P}\tau' - \mathcal{D}'\rho = \bar{\rho}\tau' + \sigma\tau - \bar{\tau}'\rho - \kappa\rho' \tag{23 b}$$

$$\mathcal{P}'\tau - \mathcal{D}\rho' = \bar{\rho}\tau + \sigma\tau' - \bar{\tau}'\rho' - \kappa\rho \tag{23 c}$$

$$\mathcal{P}\kappa' + \mathcal{D}\sigma' = \bar{\rho}(\tau' - \kappa') + \bar{\rho}'(\bar{\tau} - \bar{\tau}') + \sigma(2\tau - \bar{\tau}') + \rho(\kappa' - \kappa) - \bar{\rho}'\tau' - 2\Psi_3 \tag{23 d}$$

$$\mathcal{D}\tau' - \mathcal{D}'\tau = \rho\bar{\rho}' - \rho'\bar{\rho} \qquad (23\ e)$$

$$\mathcal{P}\rho' - \mathcal{P}'\rho = \tau\bar{\tau} - \tau'\bar{\tau}' \qquad (23\ f)$$

The set of equations (23) has been obtained by using GHP commutators (24 - I) and GHP field equations (23 - I), e.g. equation (23 a) can be obtained as follows:

Consider the commutator $[\mathcal{P}, \mathcal{D}]\phi$ and use equation (20) to get

$$[\mathcal{P}, \mathcal{D}]\phi = (\mathcal{P}\mathcal{D} - \mathcal{D}\mathcal{P})\phi = 2(\mathcal{P}\tau - \mathcal{D}\rho)\phi \qquad (24)$$

But from equations (24 b - I), (17 a) and (20), we have

$$[\mathcal{P}, \mathcal{D}]\phi = 2(\bar{\rho}\tau + \sigma\tau' - \bar{\tau}'\rho - \kappa\rho')\phi \qquad (25)$$

Equations (24) and (25) thus lead to

$$\mathcal{P}\tau - \mathcal{D}\rho = \bar{\rho}\tau + \sigma\tau' - \bar{\tau}'\rho - \kappa\rho' \qquad (26)$$

Now substituting the values of $\mathcal{P}\tau$ and $\mathcal{D}\rho$ from equations (23 c - I) and (23 d - I) in equation (26), we get after simplification

$$\mathcal{P}'\kappa - \mathcal{D}'\sigma = 2\tau'\sigma - \bar{\tau}\sigma - \bar{\rho}'\kappa - 2\Psi_1$$

which is equation (23 a).

The remaining equation of the set of equations (23) can be obtained by similar arguments.

Although the set of equations (23) apears to a complicated one but important conclusions can be made under some special choices of the spin coefficients and we have

**Theorem 1.** Let a non null electromagnetic field satisfies the source-free Maxwell's equations. Suppose it is possible to propagate the complex null tetrad along the null geodesic congruence $C(l)$ and $C(n)$ then the set of equations (23) reduces to

$$\mathcal{D}'\sigma \;=\; 2\Psi_1 \;,\; \mathcal{D}\sigma' \;=\; -2\Psi_3 \tag{27}$$

$$\mathcal{D}'\rho \;=\; 0 \;=\; \mathcal{D}\rho' \tag{28}$$

$$\rho\bar{\rho}' \;=\; \rho'\bar{\rho} \tag{29}$$

$$\mathcal{P}\rho' \;=\; \mathcal{P}'\rho \tag{30}$$

**Proof.** The hypothesis of the theorem, equation (23) and the use of Theorems 3 and 4 of Chapter II immediately lead to the proof of the theorem.

**Remark:** It may be noted that equation (27) describe the propagation of the shear of the congruence $C(l)$ and $C(n)$. The propagation of expansion and twist is given by equation (28), while equations (29) and (30) describe the coupling of the expansion and twist.

The above coupling of expansion and twist do exist even under weaker conditions as described by the following theorem.

**Theorem 2.** Let a non null electromagentic field satisfies the sourc-free Maxwell's equations and suppose that the tetrad $Z^a_\mu$ can be chosen such that $\tau$ and $\tau'$ are constant then the set of equations (23) reduces to

$$\mathcal{D}'\sigma = -\tau'\sigma + (\bar{\tau}' - \tau)\rho + \bar{\rho}'\kappa + \Psi_1 - \Phi_{01} \tag{31}$$

$$\mathcal{D}'\rho = -\bar{\rho}\tau' - \sigma\tau + \bar{\tau}'\rho + \kappa\rho' \tag{32}$$

$$\mathcal{D}\rho' = -\bar{\rho}\tau - \sigma\tau' - \bar{\tau}'\rho' + \kappa\rho \tag{33}$$

$$\mathcal{D}\sigma' = \bar{\rho}(\tau' - \kappa') + \tau'(\rho - \bar{\rho}') + \rho(\kappa' - \kappa) - \rho'\bar{\tau}' + \sigma\tau - \Psi_3 + \Phi_{21} \tag{34}$$

$$\rho\bar{\rho}' - \rho'\bar{\rho} = 0 \tag{35}$$

$$\mathcal{P}\rho' - \mathcal{P}'\rho = \tau\bar{\tau} - \tau'\bar{\tau}' \tag{36}$$

**Proof.** Since $\tau$ and $\tau'$ are constant, equation (23 c - I) leads to

$$\mathcal{P}'\kappa = (\tau - \bar{\tau}')\rho - \sigma(\bar{\tau} - \tau') - \Psi_1 - \Phi_{01}$$

Substitute it in equation (23 a) to get equation (31). As $\tau$ and $\tau'$ are constant, equations (23 b), (23 c) and (23 e) immediately lead to equations (32), (33) and (35) respectively. Equation (34) can be obtained by substituting equation (23 c' - I) in equation (23 d); while equation (36) is identically satisfied in view of the equations (23 f - I) and (23 f' - I). This completes the proof of the theorem.

## 3. Modified Lie Derivative Operator

In Chapters I and II it is seen that in GHP formalism, the operator $\Theta_a$ plays a crucial role and the use of $\Theta_a$ in place of $\nabla_a$ enable us to eliminate the spin coefficients $\alpha, \beta, \epsilon, \gamma$ which behave badly under boost-rotations. For the same reason, the modified Lie differentiation operator $L_\xi$ is defined in which $\nabla_a$ is replaced by $\Theta_a$ and thus the modified Lie derivative of a vector $u^a$ is given by

$$L_\xi \, u^a \;=\; \xi^b \Theta_b u^a \;-\; u^b \Theta_b \xi^a \tag{37}$$

Since $\Theta_a$ may be written as

$$\Theta_a \;=\; \nabla_a - p\zeta_a - q\bar{\zeta}_a \tag{38}$$

where

$$\zeta_a \;=\; \gamma l_a + \epsilon n_a - \alpha m_a - \beta \bar{m}_a \tag{39}$$

The modified Lie derivative $L_\xi$ and the Lie derivative $\mathcal{L}_\xi$ are related by

$$L_\xi \;=\; \mathcal{L}_\xi \;-\; \xi^a (p\zeta_a + q\bar{\zeta}_a) \tag{40}$$

The action of this operator on tetrad vectors may be found in [40].

In this section, we shall investigate the behaviour of the modified Lie derivative operator on the electromagnetic field tensor $F_{ij}$, the Ricci tensor $R_{ij}$ and the metric tensor $g_{ij}$ for the non null electromagntic fields.

From equations (18 - II), (38) and (40), the modified Lie derivative of $F_{ij}$ with respect to the principal null direction $l^a$ is

$$L_l \, F_{ij} \;=\; \Theta_a F_{ij} l^a \;+\; F_{ia} \Theta_j l^a \;+\; F_{aj} \Theta_i l^a \tag{41}$$

The definition of $\Theta_a$ (cf. eqn.(18 - II)) now enable us to write equation (41) as

$$
\begin{aligned}
L_l\ F_{ij} = & (l_a\mathcal{P}' + n_a\mathcal{P} - m_a\mathcal{D}' - \bar{m}_a\mathcal{D})F_{ij}l^a \\
& +F_{ia}(l_j\mathcal{P}' + n_j\mathcal{P} - m_j\mathcal{D}' - \bar{m}_j\mathcal{D})l^a \\
& +F_{aj}(l_i\mathcal{P}' + n_i\mathcal{P} - m_i\mathcal{D}' - \bar{m}_i\mathcal{D})l^a
\end{aligned} \tag{42}
$$

Using the properties of the null tetrad, equations (14 - II), (13) and (20), after lengthy calculations, equation (42) now takes the form

$$
\begin{aligned}
L_l\ F_{ij} = & 2\{\Re(\rho+\bar{\rho}) - 2\Re(\rho)l_{[i}n_{j]} + 2i\Im(\rho)m_{[i}\bar{m}_{j]}\}\phi - 2\Re\phi\{\kappa n_{[i}\bar{m}_{j]} + \bar{\kappa}n_{[i}m_{j]}\} \\
& +2i\Im\phi\{-\bar{\tau}l_{[i}m_{j]} + \tau l_{[i}\bar{m}_{j]} - 2(\rho+\bar{\rho})m_{[i}\bar{m}_{j]} + \bar{\sigma}m_{[i}m_{j]} + \sigma\bar{m}_{[i}\bar{m}_{j]}\} \\
& +2(\Re\phi + i\Im\phi)\tau' l_{[i}m_{j]} + 2(\Re\phi + i\Im\phi)\bar{\tau}' l_{[i}\bar{m}_{j]}
\end{aligned} \tag{43}
$$

which is clearly non zero for non null electromagnetic fields. However, a considerable amount of simplification results in equation (43) under the hypothesis of Theorem 1 if we consider the congruence $C(l^a)$ to be expansion-free, and we have

**Theorem 3.** Let the null geodesic congruence $C(l^a)$ and $C(n^a)$ satisfy the Maxwell's equations for a non null electromagnetic filed and the tetrad is parallely propagated along the congruences. If $C(l^a)$ is expansion-free, then

$$
L_l\ F_{ij} = -2i\Im\phi\{\ \bar{\sigma}m_{[i}m_{j]} + \sigma\bar{m}_{[i}\bar{m}_{j]}\ \} \tag{44}
$$

In the spin coefficient formalism [53], the field equations

$$
R_{ij} \;=\; -\frac{1}{4\pi}(F_{ik}F^k_j - \frac{1}{4}g_{ij}F^{rs}F_{rs})
$$

for a purely electromagnetic distribution takes the following form [4] for different types

$$\text{Type A} \;:\; R_{ij} = \chi\phi_1\bar{\phi}_1\{\, l_{(i}n_{j)} + m_{(i}\bar{m}_{j)} \,\} \tag{45 a}$$

$$\text{Type B} \;:\; R_{ij} = \frac{1}{2}\chi\phi_2\bar{\phi}_2\, l_i l_j \tag{45 b}$$

$$\text{Type C} \;:\; R_{ij} = \frac{1}{2}\chi\phi_0\bar{\phi}_0\, n_i n_j \tag{45 c}$$

It may be noted that equations (45 b) and (45 c) are the well known Lichnerowicz conditions [3] for total gravitational radiation having $l_i$ and $n_i$, respectively, as the propagation vectors.

From equations (14 - II), (45 a) and (20) it is not hard to obtain (although the calculations are very lengthy)

$$\begin{aligned} L_l\ R_{ij} = \chi\{\ & 2(\rho+\bar{\rho})l_{(i}n_{j)} + (\bar{\tau}-2\tau')l_{(i}m_{j)} + (\tau-2\bar{\tau}')l_{(i}\bar{m}_{j)} - \bar{\kappa}n_{(i}m_{j)} \\ & -\kappa n_{(i}\bar{m}_{j)} - \bar{\sigma}m_{(i}m_{j)} - \sigma\bar{m}_{(i}\bar{m}_{j)} + (\rho+\bar{\rho})m_{(i}\bar{m}_{j)}\ \}\phi\bar{\phi} \end{aligned} \tag{46}$$

for non null electromagnetic fields.

Under some special circumstances equation (46) do admit a simpler form and we have

**Theorem 4.** Let the null geodesic congruences $C(l^a)$ and $C(n^a)$ satisfy the Maxwell's equations for non null electromagnetic fields. If the tetrad is parallely propagated along $C(l^a)$ and $C(n^a)$, and $C(l^a)$ is expansion-free, then

$$L_l\ R_{ij} = -\chi\{\ \bar{\sigma}m_{(i}m_{j)} + \sigma\bar{m}_{(i}\bar{m}_{j)}\ \}\phi\bar{\phi} \tag{47}$$

which is non zero as $\sigma \neq 0$.

Finally, from the definition of modified Lie derivative, we have

$$L_l\ g_{ij} \ = \ \Theta_i l_j \ + \ \Theta_j l_i$$

which on using equation (19 - II) reduces to

$$\begin{aligned} L_l\ g_{ij} \ &= \ 2\{ \ -\bar{\tau} l_{(i}m_{j)} - \tau l_{(i}\bar{m}_{j)} - \bar{\kappa} n_{(i}m_{j)} - \kappa n_{(i}\bar{m}_{j)} \\ &+\bar{\sigma} m_{(i}m_{j)} + \sigma \bar{m}_{(i}\bar{m}_{j)} + \rho m_{(i}\bar{m}_{j)} + \bar{\rho}\bar{m}_{(i}m_{j)} \ \} \end{aligned} \tag{48}$$

so that we have

**Theorem 5.** Let the null geodesic congruences $C(l^a)$ and $C(n^a)$ satisfy the Maxwell's equations for non null electromagnetic fields and the tetrad is parallely propagated along the congruences, then

$$L_l\ g_{ij} \ = \ 2\{ \ \bar{\sigma} m_{(i}m_{j)} + \sigma \bar{m}_{(i}\bar{m}_{j)} + \rho m_{(i}\bar{m}_{j)} + \bar{\rho}\bar{m}_{(i}m_{j)} \ \} \tag{49}$$

**Remark :** It is interesting to note that for the Reissner-Nördstrom black hole [12], equations (43), (46) and (48) take the following forms respectively

$$\begin{aligned} L_l\ F_{ij} = 2\{ \ &\Re(\rho+\bar{\rho}) - 2\Re(\rho) l_{[i}n_{j]} + 2i\Im(\rho) m_{[i}\bar{m}_{j]} \ \}\phi \\ &-4i\Im\phi(\rho+\bar{\rho}) m_{[i}\bar{m}_{j]} \end{aligned} \tag{50}$$

$$L_l\ R_{ij} = \chi\ (\rho+\bar{\rho})\ \{ \ 2l_{(i}n_{j)} + m_{(i}\bar{m}_{j)} \ \}\phi\bar{\phi} \tag{51}$$

$$L_l\ g_{ij} \ = \ 2(\rho+\bar{\rho}) m_{(i}\bar{m}_{j)} \tag{52}$$

These equations (50) - (52) suggest that for the Reissner-Nördstrom black hole, the modified Lie derivatives of the electromagnetic field tensor, the Ricci tensor and the metric tensor depend on the radial coordinate (as $\rho = -1/r$) and thus for large $r$,

$$L_l\ F_{ij} = 0$$

$$L_l\ R_{ij} = 0$$

$$L_l\ g_{ij} = 0$$

Therefore, for large $r$, the Reissner-Nördstrom black hole admits Maxwell's collineation, Ricci collineation and motion.

## 4. Conclusion

In this Chapter, the non null electromagnetic fields have been studied using the compacted spin coefficient formalism. The Maxwell's equations have been translated into the language of GHP formalism (cf. equations (15), (19) - (22)). The equations describing the propagation of shear (equations (27), (31) and (34)), expansion and twist (equations (28), (32) and (33)) of the null congruences $C(l^a)$ and $C(n^a)$ associated with the non null electromagnetic fields have been obtained and the conditions (equations (29), (30) and (35) under which the expansion and twist of the congruence can be coupled together have also been given. Moreover, the propagation of shear (equation (27) is seen to be related with the longitudnal wave component of the gravitational field in $n^a$ and $l^a$ directions. The role of the modified Lie derivative operator on the electromagnetic field tensor, Ricci tensor and metric tensor has been explored. For Reissner-Nördstrom black hole these derivatives are seen to depend only on one spin coefficient $\rho$ (cf. equations (50) - (52)).

# CHAPTER IV

## General Observers, Tetrad Formalisms and Lanczos Spin Tensor

### 1. Introduction

Let $M$ be four dimensional space-time endowed with a metric $g_{ij}$ of signature $(- \ - \ - \ +)$. The curvature tensor $R^h{}_{ijk}$ is defined through the Ricci identity

$$A_{k;i;j} - A_{k;j;i} = R^h{}_{ijk} A_h \tag{1}$$

for the vector field $A_k$. The Riemann curvature tensor can be decomposed as follows:

$$R_{hijk} = C_{hijk} + E_{hijk} + G_{hijk} \tag{2}$$

where

$$E_{hijk} = \frac{1}{2}(g_{hj}S_{ik} + g_{ik}S_{hj} - g_{hk}S_{ij} - g_{ij}S_{hk}) \tag{3}$$

$$G_{hijk} = \frac{R}{12}(g_{hj}g_{ik} - g_{hk}g_{ij}) \tag{4}$$

$$S_{ij} = R_{ij} - \frac{1}{4}Rg_{ij} \tag{5}$$

$$R_{ij} = R_{ikj}{}^{k} \ , \quad R = R^i_i \tag{6}$$

The irreducible tensor $C_{hijk}$ is the Weyl tensor and satisfies the same algebraic properties as that of the Riemann tensor

$$C_{hijk} = -C_{ihjk} = -C_{hikj} = C_{jkhi}, \; C_{h[ijk]} = 0 \tag{7}$$

The other parts in the decomposition (2) have the same symmetries. Moreover,

$$C^{t}{}_{itj} = 0 , \; E^{t}{}_{itj} = S_{ij} , \; G^{t}{}_{itj} = \frac{1}{4} g_{ij} R \tag{8}$$

The Weyl tensor is completely traceless, i.e., the contraction with respect to each pair of indices vanishes, and it has ten independent components. A space-time is said to be conformally flat if $C_{hijk} = 0$.

A consequence of the equations (7) and (8) is the mixed dual property

$$^{*}C_{hijk} = C^{*}_{hijk} \tag{9}$$

where

$$^{*}C_{hijk} = \frac{1}{2}\eta_{hirs} C^{rs}{}_{jk} , \; C^{*}_{hijk} = \frac{1}{2}\eta_{jkrs} C_{hi}{}^{rs} \tag{10}$$

are, respectively, the left and right duals of the Weyl tensor.

When the Eintein's vacuum field equations

$$R_{ij} = 0 \tag{11}$$

are imposed then from equation (2) all that remains of the gravitational field is the Weyl tensor and it (Weyl tensor) describes the pure gravitational radiation field. However, when gravitational waves propagate through matter, the Weyl tensor is still pertinent.

Because the tensors $E_{hijk}$ and $G_{hijk}$ are derived from simpler irreducible tensors with fewer indices, namely $S_{ij}$ and $R$, Lanczos [43] thought that the Weyl tensor can also be derivable from a simpler tensor field, and this indeed can be done through the covariant differentiation of a tensor field $L_{ijk}$. This tensor field is now known as *Lanczos potential* and satisfies the following symmetries

$$(\text{ 40 conditions }) \quad L_{ijk} = - L_{jik} \tag{12}$$

$$(\text{ 4 conditions }) \quad L_i{}^t{}_t = 0 \quad (\text{or, } g^{kl} L_{kil} = 0) \tag{13}$$

$$(\text{ 4 conditions }) \quad L_{ijk} + L_{jki} + L_{kji} = 0 \; (\text{or, } {}^*L_i{}^t{}_t = 0) \tag{14}$$

In this way the tensor field $L_{ijk}$, which has atmost sixtyfour independent components, has been reduced to atmost sixteen independent components. In order to have a perfect match with the Weyl tensor, Lanczos imposed the six equations

$$L_{ij}{}^k{}_{;k} = 0 \tag{15}$$

so that $L_{ijk}$ is a field with only ten effective degrees of freedom. Equation (15) is known as *Lanczos differential gauge conditions* and is equivalent to

$$( L^k_{ij} + L_{ij}{}^{jk} )_{;k} = 0 \tag{15 a}$$

The Weyl tensor $C_{hijk}$ is generated by $L_{ijk}$ through the equation ([14], [15], [16]; throughout this Chapter, we shall use the notations of reference [16])

$$\begin{aligned} C_{hijk} &= L_{hij;k} - L_{hik;j} + L_{jkh;i} - L_{jki;h} \\ &+ L_{(hk)} g_{ij} + L_{(ij)} g_{hk} - L_{(hj)} g_{ik} - L_{(ik)} g_{hj} \\ &+ \frac{2}{3} L^{pq}{}_{p;q} ( g_{hj} g_{ik} - g_{hk} g_{ij} ) \end{aligned} \tag{16}$$

where

$$L_{ij} = L_i{}^k{}_{j;k} - L_i{}^k{}_{k;j} \tag{17}$$

The potential field relations of the Lanczos potential to the Weyl tensor as given by equation (16) are known as *Weyl-Lanczos relations.*

From equations (13), (15) and (17), the Weyl-Lanczos relations (16) can also be expressed as ([14])

$$\begin{aligned} C_{hijk} &= L_{hij;k} - L_{hik;j} + L_{jkh;i} - L_{jki;h} \\ &+\frac{1}{2}\left( L_i{}^p{}_{j;p} + L_j{}^p{}_{i;p} \right) g_{hk} + \frac{1}{2}\left( L_h{}^p{}_{k;p} + L_k{}^p{}_{h;p} \right) g_{ij} \\ &-\frac{1}{2}\left( L_h{}^p{}_{j;p} + L_j{}^p{}_{h;p} \right) g_{ik} - \frac{1}{2}\left( L_i{}^p{}_{k;p} + L_k{}^p{}_{i;p} \right) g_{hj} \end{aligned} \tag{18}$$

The introduction of the potential tensor $L_{ijk}$ for the Weyl tensor $C_{hijk}$ enabled Lanczos to achieve the gravitational analogue of the potential four vector $A_i$ for the electromagnetic field

$$F_{ij} = A_{i;j} - A_{j;i}$$

(For the sake of completeness, a comparison between the analogies of the electromagnetic and gravitational theories is made in the Appendix to the thesis).

It is known that twenty Bianchi identities

$$R_{hijk;l} + R_{hikl;j} + R_{hilj;k} = 0 \tag{19}$$

decompose into four irreducible equations

$$G^{ik}{}_{;k} = 0 \tag{20}$$

where

$$G_{ij} = R_{ij} - \frac{1}{2} R\, g_{ij} \tag{21}$$

is the Einstein tensor. Moreover, we have sixteen irreducible equations

$$C_{ijk}{}^{l}{}_{;l} = J_{ijk} \tag{22}$$

where

$$J_{ijk} = R_{k[i;j]} - \frac{1}{6} g_{k[i} R_{;j]} \tag{23}$$

is known as the Schouten tensor.

These equations suggest that like potential field $L_{ijk}$, the tensor field $J_{ijk}$ is irreducible with atmost sixteen independent components because of its symmetries and only ten effective degrees of freedom because of the six identities

$$J_{ij}{}^{k}{}_{;k} = 0 \tag{24}$$

It may be noted that the right hand side of equation (23) can be expressed entirely in terms of covariant derivatives of the irreducible tensor field $S_{ij}$, i.e., the traceless part of Ricci and Einstein tensors.
Using equations (16) and (22), Dolan and Kim [14] obtained the following wave equation

$$\begin{aligned} \Box\, L_{ijk} + R^{t}{}_{ipk} L_{j}{}^{p}{}_{t} - R^{t}{}_{jkp} L_{i}{}^{p}{}_{t} - R^{t}{}_{ijp} L_{t}{}^{p}{}_{k} + R_{tk} L_{ij}{}^{t} \\ - R_{tj} L_{k}{}^{t}{}_{i} + R_{ti} L_{k}{}^{t}{}_{j} + R^{q}{}_{jtp} L_{q}{}^{tp} g_{ik} - R^{q}{}_{itp} L_{q}{}^{tp} g_{jk} = J_{ijk} \end{aligned} \tag{25}$$

which on using the identity ([14])

$$C^{t}{}_{ipk} L_{j}{}^{p}{}_{t} - C^{t}{}_{jpk} L_{i}{}^{p}{}_{t} - C^{t}{}_{ijp} L_{t}{}^{p}{}_{k} + C^{r}{}_{jtp} L_{r}{}^{tp} g_{ik} - C^{r}{}_{itp} L_{r}{}^{tp} g_{jk} = 0 \tag{26}$$

takes the form

$$\Box\, L_{ijk} + 2R^t{}_k L_{ijt} - R_i{}^t L_{jkt} + R_j{}^t L_{ikt} - g_{ik} R^{pt} L_{pjt}$$
$$+ g_{jk} R^{pt} L_{pit} - \frac{1}{2} R L_{ijk} = J_{ijk} \tag{27}$$

so that in empty space there results a remarkably simple wave equation

$$\Box\, L_{ijk} = 0 \tag{28}$$

Although the existence of a tensor $L_{ijk}$ as a potential to the Weyl tensor $C_{hijk}$ was establish by C. Lanczos in 1962, there was a little devalopment in the subject for quite some time. Using the spinor calculus a study of the Lanczos potential was made by Zund [60], while the existence of Lanczos potential to a larger class of 4-potential and to a larger class of 3-tensors was proved by Bampi and Caviglia [9]. Later on using the spinor formalism, Illge [36] proved the existence of Lanczos potential in four dimensions and obtained the wave equations for the Lanczos potential both in spinor and tensor forms. These wave equations were further studied by Dolan and Kim [14] who gave a correct version of the wave equation appeared in the paper of Illge [36]. A number of identities satisfied by the 3-tensor $L_{ijk}$ was established by Edgar [19] and Edgar and Höglund [21]. An algorithm for calculating the Lanczos potential for perfect fluid space-times, under certain conditions, was proposed by Novello and Velloso [54]. Dolan and Kim [15] not only generalised the results of Novello and Velloso but also made a correction in the earlier versions of the Weyl-Lanczos equations appeared in the litrature (cf. Zund [60] and Ares de Parga et al [7]). The gauge conditions in a more geometric setting were studied by Hammon and Norris [31]. For a larger class of space-times, using spinor formalism, the Lanczos potential was obtained by Torris de Castillo [59], while using the Newman-Penrose formalism, Lopez-Bonilla, Ares de Parga and co-workers, in a series of papers ([7], [8], [11], [28], [29], [45]) obtained the Lanczos potential for various algebraically special space-times. Massa and Pagani [52] have shown that in four dimensions the Riemann tensor can not, in general, be expressed in terms of a Lanczos potential. This result was later on generalised by Edgar [20] for $n$ dimensions. The Lanczos potential in Kerr geometry has been studied by Bergqvist [10], and a relationship between the Lanczos potential and the Ernst potential has been established by Dolan and Muratori [16]. More recently, Andersson and Edgar [6] showed that the Lanczos potential can be defined in a very simple way directly from the spinor dyad and has obtained some links between the Lanczos potential and the spin coefficients for some space-times, while Calva, Lopez-Bonilla and Ovando [11] obtained a relationship between the spin coefficients and the Lanczos scalars for some special space-times ( For a more comprehensive and updated review, upto 1997, the reader is referred to the paper of Edgar and Höglund [21]).

Motivated by the above disscussion about the importance of the Lanczos potential in general relativity, the present Chapter is devoted to a study of this tensor. In section 2, the method of general observers has been considered. The kinametical quantities, expansion, shear, twist, etc. and the equations satisfied by them have been translated in the language of the spin coefficient formalism. The tensorial versions of the earlier results of Novello and Velloso [54], as described by Dolan and Kim [15], about the Lanczos potentials for several space-times have been written in terms of the spin coefficients. The conjecture of Lopez-Bonilla and co-workers that there is some linear relationship between the spin coefficients and Lanczos scalars has been verified for the perfect fluid space-times and in the process we have obtained the Lanczos potential for the Gödel solution. For a given geometry, the construction of $L_{ijk}$ is equivalent to solving equation (16) with equations (13), (14) and (15) as constraints; and as seen from the above disscusion that there are several ways of solving equation (16) although none of them are as straight forward as one would like them to be. Section 3 of this Chapter contains yet another attempt to solve Weyl-Lanczos relations (16). These relations alongwith the Lanczos differential gauge conditions (15) have been obtained in terms of the compacted spin coefficient formalism ( cf. Chapter I) and a potential for a Petrov type D space-time has been found. These results are then applied to a Kerr black hole. Finally, a disscusion of the results of this Chapter is given in section 4.

## 2. General Observers and Lanczos Potential

For a gravitational field with perfect fluid source, the basic covariant variables are : the fluid scalars $\theta$ (expansion), $\tilde{\rho}$ (energy density), $p$ (pressure);

the fluid spatial vectors $\dot{u}_i$ (4-acceleration), $\omega_i$ (vorticity); the spatial trace-free symmetric tensors $\sigma_{ij}$ (fluid shear), the electric ($E_{ij}$) and the magnetic ($H_{ij}$) parts of the Weyl tensor; and the projection tensor $h_{ij}$ which projects orthogonal to the fluid 4-velocity vector $u_i$.

These quantities, for a unit timelike vector field $u_i$ such that $u_i u^i = 1$ (physically, the timelike vector field $u_i$ is often taken to be the 4-velocity of the fluid), are defined as follows ([26])

(i) The *projection tensor* $h_{ij}$ is defined as

$$h_{ij} = g_{ij} - u_i u_j \tag{29 a}$$

and has the following properties:

$$h_{ij} = h_{ji} \, , \; h_i{}^k h_{kj} = h_{ij} \, , \; h^i{}_i = 3 \, , \; h_{ij} u^j = 0 \tag{29 b}$$

(ii) The *expansion scalar* $\theta$ is defined as

$$\theta = u^i{}_{;i} \tag{30}$$

(iii) The *acceleration vector* $a_i$ is given by

$$a_i = \dot{u}_i = u_{i;j} \, u^j \tag{31 a}$$

and is such that

$$a_i \, u^i = 0 \tag{31 b}$$

(iv) The symmetric *shear tensor* $\sigma_{ij}$ is defined by

$$\sigma_{ij} = h_i{}^k h_j{}^l \, u_{(k;l)} - \frac{1}{3}\theta \, h_{ij} \tag{32 a}$$

and has the properties

$$\sigma_{ij}\, u^j = 0 \, , \; \sigma_i{}^i = 0 \tag{32 b}$$

(v) The antisymmetric *vorticity or rotation tensor* $\omega_{ij}$ is given by

$$\omega_{ij} = h_i{}^k h_j{}^l \, u_{[k;l]} \tag{33 a}$$

This rotation tensor satisfies the equations

$$\omega_{ij}\, u^i = \omega_{ij}\, u^j = 0 \tag{33 b}$$

and is equivalent to a vorticity vector

$$\omega^i = \frac{1}{2}\eta^{ijkl}\, \omega_{jk}\, u_l \tag{33 c}$$

so that

$$\omega_{ij} = \eta_{ijkl}\, \omega^k\, u^l \tag{33 d}$$

where $\eta^{ijkl}$ is completely anti-symmetric Levi-Civita tensor.

It may be noted from equations (31 b), (32 b) and (33 b) that $a_i$, $\sigma_{ij}$ and $\omega_{ij}$ are spacelike.

(vi) The *electric* and *magnetic* parts of the Weyl tensor as measured by an observer with a timelike 4-velocity vector $u^i$ are defined, respectively, as (see also [5])

$$E_{ik} = C_{ijkl}\, u^j\, u^l \tag{34}$$

$$H_{ik} = {}^*C_{ijkl}\, u^j\, u^l = \frac{1}{2}\eta_{ij}{}^{mn} C_{mnkl}\, u^j\, u^l \tag{35}$$

Here both $E_{ik}$ and $H_{ik}$ are spacelike, symmetric and traceless, i.e.,

$$E_{ik} = E_{ki} \ , \ E_{ik}\, u^k = 0 \ , \ E_{ik} g^{ik} = E^t{}_t = 0 \tag{36 a}$$

$$H_{ik} = H_{ki} \ , \ H_{ik}\, u^k = 0 \ , \ H_{ik} g^{ik} = H^t{}_t = 0 \tag{36 b}$$

The Weyl tensor is said to be purely electric if $H_{ik} = 0$ and purely magnetic if $E_{ik} = 0$ and in terms of $E_{ik}$ and $H_{ik}$, the Weyl tensor can be decomposed as

$$\begin{aligned} C_{hijk} &= 2u_h u_j E_{ik} + 2u_i u_k E_{hj} - 2u_h u_k E_{ij} - 2u_i u_j E_{hk} \\ &\quad + g_{hk} E_{ij} + g_{ij} E_{hk} - g_{hj} E_{ik} - g_{ik} E_{hj} \\ &+ \eta_{hi}{}^{pq}\, u_p\, u_q\, H_{qj} - \eta_{hi}{}^{pq}\, u_p\, u_j\, H_{qk} + \eta_{jk}{}^{pq}\, u_i\, u_p\, H_{hq} - \eta_{jk}{}^{pq}\, u_h\, u_p\, H_{iq} \end{aligned} \tag{37}$$

**(vii)** The covariant derivative of $u_i$ may be decomposed into its irreducible parts

$$u_{i;j} = \sigma_{ij} + \frac{1}{3}\theta\, h_{ij} + \omega_{ij} + a_i\, u_j \tag{38}$$

where $h_{ij}, \theta, a_i, \sigma_{ij}$ and $\omega_{ij}$ are, respectively, defined through (i) - (v).

**(viii)** The energy density $\tilde{\rho}$ and the pressure $p$ are given by the energy momentum tensor $T_{ij}$ of the perfect fluid

$$T_{ij} = \tilde{\rho}\, u_i\, u_j - p\, h_{ij} \tag{39}$$

The relativistic equations of the conservation of energy and momentum are

$$T^{ij}{}_{;j} = 0 \tag{40}$$

In this section, we shall obtain the kinematical quantities and the equations satisfied by them in terms of the spin coefficient formalism of Newman and Penrose [53]. These results will then be used to find the Lanczos potential for the perfect fluid space-times.

Let $Z^i_\mu = \{ l^i, n^i, m^i, \bar{m}^i \}$ be a complex null tetrad satisfying the properties ( 1-3 - I). The metric tensor $g_{ij}$, in terms of the tetrad vectors, can be written as

$$g_{ij} = l_i n_j + n_i l_j - m_i \bar{m}_j - \bar{m}_i m_j \tag{41}$$

We choose the 4-velocity vector $u^i$ as

$$u^i = \frac{1}{\sqrt{2}} ( l^i + n^i ) \tag{42 a}$$

such that

$$u_i \, u^i = 1 \tag{42 b}$$

From equations (41) and (42), the projection tensor $h_{ij}$ (as defined by equation (29 a)) takes the form

$$h_{ij} = l_{(i} n_{j)} - 2 m_{(i} \bar{m}_{j)} - \frac{1}{2} ( l_i l_j + n_i n_j ) \tag{43}$$

By virtue of this equation and the properties of the tetrad vectors, condition (29 b) can easily be verified.

The properties of the tetrad vectors also lead to the following relations (see also Appendix to the thesis)

$$l_{i;j} \;=\; (\gamma+\bar{\gamma})l_j l_i \;-\; \bar{\tau} l_j m_i \;-\; \tau l_j \bar{m}_i \;+\; (\epsilon+\bar{\epsilon})n_j l_i \;-\; \bar{\kappa} n_j m_i \;-\; \kappa n_j \bar{m}_i$$
$$-\,(\alpha+\bar{\beta})m_j l_i + \bar{\sigma} m_j m_i + \rho m_j \bar{m}_i - (\bar{\alpha}+\beta)\bar{m}_j l_i + \bar{\rho}\bar{m}_j m_i + \sigma \bar{m}_j \bar{m}_i \quad (44)$$

$$n_{i;j} \;=\; -(\gamma+\bar{\gamma})l_j n_i \;-\; \nu l_j m_i \;+\; \bar{\nu} l_j \bar{m}_i \;-\; (\epsilon+\bar{\epsilon})n_j n_i \;+\; \pi n_j m_i \;+\; \bar{\pi} n_j \bar{m}_i$$
$$+\,(\alpha+\bar{\beta})m_j n_i - \lambda m_j m_i - \bar{\mu} m_j \bar{m}_i + (\bar{\alpha}+\beta)\bar{m}_j n_i - \mu \bar{m}_j m_i - \bar{\lambda}\bar{m}_j \bar{m}_i \quad (45)$$

Also we have

$$m_{i;j} \;=\; \bar{\nu} l_j l_i - \tau l_j n_i + (\gamma - \bar{\gamma}) l_j m_i + \bar{\pi} n_j l_i - \kappa n_j n_i + (\epsilon - \bar{\epsilon}) n_j m_i$$
$$-\bar{\mu} m_j l_i + \rho m_j n_i + (\bar{\beta} - \alpha) m_j m_i - \bar{\lambda} \bar{m}_j l_i + \sigma \bar{m}_j n_i + (\bar{\alpha} - \beta)\bar{m}_j m_i \quad (45\ a)$$

The contractions of equations (44) and (45) lead to

$$l^i{}_{;i} \;=\; (\epsilon+\bar{\epsilon}) \;-\; (\rho+\bar{\rho}) \tag{46}$$

$$n^i{}_{;i} \;=\; (\mu+\bar{\mu}) \;-\; (\gamma+\bar{\gamma}) \tag{47}$$

Moreover, from equations (45) and (46), we have (see also eqns. (105 a,b - App.) and (106 a,b - App.))

$$l_{i;j}\, l^j \;=\; (\epsilon+\bar{\epsilon})\, l_i \;-\; \bar{\kappa} m_i \;-\; \kappa \bar{m}_i \tag{48}$$

$$l_{i;j}\, n^j \;=\; (\gamma+\bar{\gamma})\, l_i \;-\; \bar{\tau} m_i \;-\; \tau \bar{m}_i \tag{49}$$

$$n_{i;j}\, l^j \;=\; -(\epsilon+\bar{\epsilon})\, n_i \;+\; \pi m_i \;+\; \bar{\pi} \bar{m}_i \tag{50}$$

$$n_{i;j}\, n^j \;=\; -(\gamma+\bar{\gamma})\, n_i \;+\; \nu m_i \;+\; \bar{\nu} \bar{m}_i \tag{51}$$

while from equations (42 a), (44) and (45) we have

$$
\begin{aligned}
u_{i;j} \;=\; \frac{1}{\sqrt{2}} \{\; & (\gamma+\bar{\gamma})l_j l_i \;-\; \bar{\tau} l_j m_i \;-\; \tau l_j \bar{m}_i \;+\; (\epsilon+\bar{\epsilon}) n_j l_i \\
& - \bar{\kappa} n_j m_i \;-\; \kappa n_j \bar{m}_i \;-\; (\alpha+\bar{\beta}) m_j l_i \;+\; \bar{\sigma} m_j m_i \\
& + \rho m_j \bar{m}_i \;-\; (\bar{\alpha}+\beta)\bar{m}_j l_i \;+\; \bar{\rho}\bar{m}_j m_i \;+\; \sigma \bar{m}_j \bar{m}_i \\
& - (\gamma+\bar{\gamma}) l_j n_i \;+\; \nu l_j m_i \;+\; \bar{\nu} l_j \bar{n}_i \;-\; (\epsilon+\bar{\epsilon}) n_j n_i \\
& + \pi n_j m_i \;+\; \bar{\pi} n_j \bar{m}_i \;+\; (\alpha+\bar{\beta}) m_j n_i \;-\; \lambda m_j m_i \\
& \bar{\mu} m_j \bar{m}_i \;+\; (\bar{\alpha}+\beta)\bar{m}_j n_i \;-\; \mu \bar{m}_j m_i \;-\; \bar{\lambda}\bar{m}_j \bar{m}_i \;\}
\end{aligned}
\tag{52}
$$

so that

$$
\begin{aligned}
u_{(i;j)} \;=\; \frac{1}{\sqrt{2}} \{\; & (\gamma+\bar{\gamma})\; l_{(j} l_{i)} \;+\; (\epsilon+\bar{\epsilon}-\gamma-\bar{\gamma})\; n_{(j} l_{i)} \\
& + (\epsilon+\bar{\epsilon})\; n_{(j} n_{i)} \;-\; (\bar{\kappa}-\pi-\alpha-\bar{\beta})\; n_{(j} m_{i)} \;-\; (\bar{\tau}+\alpha+\bar{\beta}-\nu)\; l_{(j} m_{i)} \\
& + (\bar{\sigma}-\lambda)\; m_{(j} m_{i)} \;+\; (\rho+\bar{\rho}-\mu-\bar{\mu})\; m_{(j}\bar{m}_{i)} \;-\; (\tau+\bar{\alpha}+\beta-\bar{\nu})\; l_{(j}\bar{m}_{i)} \\
& - (\kappa-\bar{\pi}-\bar{\alpha}-\beta)\; n_{(j}\bar{m}_{i)} \;+\; (\sigma-\bar{\lambda})\; \bar{m}_{(j}\bar{m}_{i)} \;\}
\end{aligned}
\tag{53}
$$

and

$$
\begin{aligned}
u_{[i;j]} \;=\; \frac{1}{\sqrt{2}} \{\; & (\epsilon+\bar{\epsilon}+\gamma+\bar{\gamma})\; n_{[j} l_{i]} \;+\; (\alpha+\bar{\beta}-\bar{\tau}+\nu)\; l_{[j} m_{i]} \\
& + (\bar{\alpha}+\beta-\tau+\bar{\nu})\; l_{[j}\bar{m}_{i]} \;-\; (\bar{\kappa}-\pi+\alpha+\bar{\beta})\; n_{[j} m_{i]} \\
& - (\kappa-\bar{\pi}+\bar{\alpha}+\beta)\; n_{[j}\bar{m}_{i]} \;+\; (\rho-\bar{\rho}+\mu-\bar{\mu})\; m_{[j}\bar{m}_{i]} \;\}
\end{aligned}
\tag{54}
$$

Thus, from equations (30), (46) and (47), the expansion $\theta$ in terms of the spin coefficients can be expressed as

$$
\theta \;=\; \frac{1}{\sqrt{2}} \left(\; \epsilon+\bar{\epsilon}-\gamma-\bar{\gamma}-\rho-\bar{\rho}+\mu+\bar{\mu} \right)
\tag{55}
$$

The acceleration vector $a_i$, from equations (31 a) and (52), in terms of the spin coefficients is given by

$$a_i = \frac{1}{\sqrt{2}} ( \epsilon + \bar{\epsilon} + \gamma + \bar{\gamma}) v_i$$

$$+ \frac{1}{2} \{ (\nu - \pi - \bar{\kappa} - \bar{\tau}) m_i + (\bar{\nu} - \bar{\pi} - \kappa - \tau) \bar{m}_i \} \tag{56}$$

where the vector $v_i$ is defined as

$$v_i = \frac{1}{\sqrt{2}} ( l_i - n_i ) \tag{57 a}$$

and is such that

$$v_i v^i = -1 , \ v_i u^i = 0 \tag{57 b}$$

The symmetric shear tensor $\sigma_{ij}$ (defined by equation (32)) using equations (43), (53) and (55), after simplification can be expressed in terms of the spin coefficients by the following equation

$$\sigma_{ij} = \frac{1}{3\sqrt{2}} \{ 2(\epsilon+\bar{\epsilon})+\rho+\bar{\rho}-2(\gamma+\bar{\gamma})-\mu-\bar{\mu} \} (v_i v_j - \frac{1}{2} m_i \bar{m}_j - \frac{1}{2} \bar{m}_i m_j)$$

$$+ \frac{1}{4} \{ [2(\alpha + \bar{\beta}) + \pi + \bar{\tau} - \nu - \bar{\kappa}] (v_i m_j + v_j m_i) \}$$

$$+ \frac{1}{4} \{[2(\bar{\alpha} + \beta) + \bar{\pi} + \tau - \bar{\nu} - \kappa] (v_i \bar{m}_j + v_j \bar{m}_i) \}$$

$$+ \frac{1}{2\sqrt{2}} \{ (\lambda - \bar{\sigma}) m_i m_j + (\bar{\lambda} - \sigma) \bar{m}_i \bar{m}_j \} \tag{58}$$

While using equations (43) and (54), the rotation tensor (defined by equation (33 a)), in terms of the spin coefficients, can be written as

$$
\begin{aligned}
\omega_{ij} &= \frac{1}{4}\{\ [2(\alpha+\bar\beta)+\nu+\bar\kappa-\pi-\bar\tau]\ (v_i m_j - v_j m_i)\ \} \\
&+ \frac{1}{4}\{\ [2(\bar\alpha+\beta)+\bar\nu+\kappa-\bar\pi-\tau]\ (v_i \bar m_j - v_j \bar m_i)\ \} \\
&+ \frac{1}{2\sqrt{2}}\{\ \rho-\bar\rho+\mu-\bar\mu\ \}\ (m_i \bar m_j - \bar m_i m_j)
\end{aligned} \tag{59}
$$

It is interesting to note that from equations (42), (43) and (55) - (59), the covariant derivative of $u_i$ (given by equation (52)) can be expressed as

$$
u_{i;j} = \frac{1}{3}\,\theta\, h_{ij} + a_i\, u_j + \sigma_{ij} + \omega_{ij} \tag{60}
$$

which is same as equation (38).

The conservation of energy and momentum (cf. equation (40)) together with equation (39) yield

$$
\dot{\tilde\rho} + (\ \tilde\rho + p\ )\,\theta = 0 \tag{61}
$$

and

$$
(\ \tilde\rho + p\ )\, a^i - p_{;j}\, h^{ij} = 0 \tag{62}
$$

The momentum conservation equation (62) shows that the acceleration $a_i$ of the fluid is determined by the spatial pressure gradiant and we have

$$
a^i = \frac{p_{;j}\, h^{ij}}{\tilde\rho + p} \tag{62 a}
$$

From equations (105 - App.), (106 - App.), the properties of the tetrad and equation (55), the Newman-Penrose version of equation (61) is given by

$$(D + D')\tilde{\rho} \;+\; \sqrt{2}\,(2\tilde{\rho} + p)\,(\epsilon + \bar{\epsilon} - \gamma - \bar{\gamma} - \rho - \bar{\rho} + \mu + \bar{\mu}) \;=\; 0 \qquad (63)$$

The definition of the projection tensor enable us to write equation (62) as

$$(\,\tilde{\rho} \;+\; p\,)\,a^i \;-\; p_{;j}\,(\,g^{ij} \;-\; u^i u^j\,) \;=\; 0 \qquad (62\;b)$$

which on using equation (109 - App.) takes the form

$$(\tilde{\rho} + p)\;a_i \;-\; (l_j D'p + n_j Dp - \bar{m}_j \delta p - m_j \delta' p)\;(g^{ij} - u^i u^j) \;=\; 0 \qquad (64)$$

Now applying equations (41), (56), (1 -I) - (3 -I) and the linear dependence of the tetrad vectors $l^i, n^i, m^i, \bar{m}^i$ in equation (64) it can easily be shown that the spin coefficient version of equation (62) is equivalent to the following set of equations

$$Dp \;-\; D'p \;=\; -\,(\,\tilde{\rho} + p\,)\,(\,\epsilon + \bar{\epsilon} + \gamma + \bar{\gamma}\,) \qquad (65)$$

$$\delta p \;=\; -\frac{1}{2}\,(\,\tilde{\rho} + p\,)\,(\,\bar{\nu} - \bar{\pi} - \kappa - \tau\,) \qquad (66)$$

$$\delta' p \;=\; -\frac{1}{2}\,(\,\tilde{\rho} + p\,)\,(\,\nu - \pi - \bar{\kappa} - \bar{\tau}\,) \qquad (67)$$

From the definitions of $u^i$ and $v^i$ and the properties of the tetrad $Z^a_\mu$ we also have

$$u_i\, l^i \;=\; u_i\, n^i \;=\; \frac{1}{\sqrt{2}}\;,\; u_i\, m^i \;=\; u_i\, \bar{m}^i \;=\; 0 \qquad (68\;a)$$

$$-v_i\, l^i \;=\; v_i\, n^i \;=\; \frac{1}{\sqrt{2}}\;,\; v_i\, m^i \;=\; v_i\, \bar{m}^i \;=\; 0 \qquad (68\;b)$$

$$g_{ij}\, l^j \;=\; l_i\;,\; g_{ij}\, n^j \;=\; n_i\;,\; g_{ij}\, m^j \;=\; m_i\;,\; g_{ij}\, \bar{m}^j \;=\; \bar{m}_i \qquad (68\;c)$$

$$g_{ij}\, l^i\, l^j \;=\; g_{ij}\, n^i\, n^j \;=\; g_{ij}\, m^i\, m^j \;=\; g_{ij}\, l^i\, m^j \;=\; g_{ij}\, l^i\, \bar{m}^j \;=\; g_{ij}\, n^i\, m^j$$

$$=\; g_{ij}\, n^i\, \bar{m}^j \;=\; g_{ij}\, m^i\, n^j \;=\; g_{ij}\, \bar{m}^i\, n^j \;=\; g_{ij}\, \bar{m}^i\, \bar{m}^j \;=\; 0 \qquad (68\ d)$$

$$g_{ij}\, l^i\, n^j \;=\; g_{ij}\, n^i\, l^j \;=\; g_{ij}\, m^i\, \bar{m}^j \;=\; g_{ij}\, \bar{m}^i\, m^j \;=\; 1 \qquad (68\ e)$$

so that equation (56) leads to

$$a_i\, l^i \;=\; -\frac{1}{2}\,(\, \epsilon + \bar{\epsilon} + \gamma + \bar{\gamma}\,) \qquad (69\ a)$$

$$a_i\, n^i \;=\; \frac{1}{2}\,(\, \epsilon + \bar{\epsilon} + \gamma + \bar{\gamma}\,) \qquad (69\ b)$$

$$a_i\, m^i \;=\; -\frac{1}{2}\,(\, \bar{\nu} - \bar{\pi} - \kappa - \tau\,) \qquad (69\ c)$$

$$a_i\, \bar{m}^i \;=\; \frac{1}{2}\,(\, \nu - \pi - \bar{\kappa} - \bar{\tau}\,) \qquad (69\ d)$$

Also, from equations (58) and (68) it can easily be verified that

$$\sigma_{ij}\, l^j \;=\; -\,\sigma_{ij}\, n^j \;=\; -\frac{1}{6}\,\{\, 2(\epsilon+\bar{\epsilon}) + \rho + \bar{\rho} - 2(\gamma+\bar{\gamma}) - \mu - \bar{\mu}\,\}\, v_i$$

$$-\frac{1}{4\sqrt{2}}\,[\,\{2(\alpha+\bar{\beta})+\pi+\bar{\tau}-\nu-\bar{\kappa}\,\}m_i \,+\, \{2(\bar{\alpha}+\beta)+\bar{\pi}+\tau-\bar{\nu}-\kappa\,\}\bar{m}_i\,] \quad (70\ a)$$

$$\sigma_{ij}\, m^j \;=\; \frac{1}{6\sqrt{2}}\,\{\, 2(\epsilon+\bar{\epsilon}) + \rho + \bar{\rho} - 2(\gamma+\bar{\gamma}) - \mu - \bar{\mu}\,\}\, m_i$$

$$-\frac{1}{4}\,\{2(\bar{\alpha}+\beta) + \bar{\pi} + \tau - \bar{\nu} - \kappa\,\}v_i \;-\; \frac{1}{2\sqrt{2}}\,(\bar{\lambda} - \sigma)\,\bar{m}_i \qquad (70\ b)$$

$$\sigma_{ij}\ \bar{m}^j = \frac{1}{6\sqrt{2}}\ \{\ 2(\epsilon+\bar{\epsilon})+\rho+\bar{\rho}-2(\gamma+\bar{\gamma})-\mu-\bar{\mu}\ \}\ \bar{m}_i$$

$$-\frac{1}{4}\ \{2(\alpha+\bar{\beta})+\pi+\bar{\tau}-\nu-\bar{\kappa}\ \}v_i\ -\frac{1}{2\sqrt{2}}\ (\lambda-\bar{\sigma})\ m_i \tag{70 c}$$

$$\sigma_{ij}\ l^i\ l^j\ =\ -\sigma_{ij}\ n^i\ l^j\ =\ \sigma_{ij}\ n^i\ n^j\ =-\ \sigma_{ij}\ \bar{m}^i\ m^j$$
$$= \frac{1}{6\sqrt{2}}\ \{\ 2(\epsilon+\bar{\epsilon})+\rho+\bar{\rho}-2(\gamma+\bar{\gamma})-\mu-\bar{\mu}\ \} \tag{70 d}$$

$$\sigma_{ij}\ m^i\ l^j\ =\ -\sigma_{ij}\ m^i\ n^j\ =\ \frac{1}{4\sqrt{2}}\ \{\ 2(\bar{\alpha}+\beta)+\bar{\pi}+\tau-\bar{\nu}-\kappa\ \} \tag{70 e}$$

$$\sigma_{ij}\ \bar{m}^i\ l^j\ =\ -\sigma_{ij}\ \bar{m}^i\ n^j\ =\ \frac{1}{4\sqrt{2}}\ \{\ 2(\alpha+\bar{\beta})+\pi+\bar{\tau}-\nu-\bar{\kappa}\ \} \tag{70 f}$$

$$\sigma_{ij}\ m^i\ m^j\ =\ \frac{1}{2\sqrt{2}}\ (\bar{\lambda}-\sigma) \tag{70 g}$$

$$\sigma_{ij}\ \bar{m}^i\ \bar{m}^j\ =\ \frac{1}{2\sqrt{2}}\ (\lambda-\bar{\sigma}) \tag{70 h}$$

While, equations (59) and (68) lead to the following relations

$$\omega_{ij}\ l^j\ =\ \omega_{ij}\ n^j\ =\ \frac{1}{4\sqrt{2}}\ [\ \{\ 2(\alpha+\bar{\beta})+\nu+\bar{\kappa}-\pi-\bar{\tau}\ \}\ m_i$$
$$+\ \{\ 2(\bar{\alpha}+\beta)+\bar{\nu}+\kappa-\bar{\pi}-\tau\ \}\ ]\ \bar{m}_i \tag{71 a}$$

$$\omega_{ij}\ m^j\ =\ -\frac{1}{4}\ \{\ 2(\bar{\alpha}+\beta)+\bar{\nu}+\kappa-\bar{\pi}-\tau\ \}\ v_i \tag{71 b}$$

$$\omega_{ij}\ \bar{m}^j\ =\ -\frac{1}{4}\ \{\ 2(\alpha+\bar{\beta})+\nu+\bar{\kappa}-\pi-\bar{\tau}\ \}\ v_i \tag{71 c}$$

$$\omega_{ij}\, m^i\, l^j \;=\; -\,\omega_{ij}\, m^i\, n^j \;=\; -\frac{1}{4\sqrt{2}}\,\{\, 2(\bar{\alpha}+\beta)+\bar{\nu}+\kappa-\bar{\pi}-\tau \,\} \quad (71\ d)$$

$$\omega_{ij}\, \bar{m}^i\, l^j \;=\; \omega_{ij}\, \bar{m}^i\, n^j \;=\; \frac{1}{4\sqrt{2}}\,\{\, 2(\alpha+\bar{\beta})+\nu+\bar{\kappa}-\pi-\bar{\tau} \,\} \qquad (71\ e)$$

$$\begin{aligned}\omega_{ij}\, l^i\, l^j \;&=\; \omega_{ij}\, n^i\, l^j \;=\; \omega_{ij}\, n^i\, n^j \;=\; \omega_{ij}\, m^i\, m^j \\ &=\; \omega_{ij}\, \bar{m}^i\, m^j \;=\; \omega_{ij}\, \bar{m}^i\, \bar{m}^j \;=\; 0 \end{aligned} \qquad (71\ f)$$

From equations (55), (58) and (59) it may be noted that the vanishing of expansion, shear and rotation (i.e., $\theta = 0, \sigma_{ij} = 0, \omega_{ij} = 0$) give rise to the following relations among the spin coefficients

$$\epsilon + \bar{\epsilon} = -(\gamma + \bar{\gamma})\,,\ \lambda - \bar{\sigma} = 0$$

$$\alpha + \bar{\beta} = 0\,,\ \pi + \bar{\tau} - \nu - \bar{\kappa} = 0\,,\ \bar{\mu} - \rho = 0 \qquad (72)$$

The above discussions now allow us to rewrite the results of Novello and Velloso [54] (as described Dolan and Muratori [16]) in terms of the spin coefficients as follows:

**Lemma 1.** If in a given space-time there is a field of observers $u^i$ that is shear-free, irrotational and expansion-free, then the Lanczos potential is given by

$$\begin{aligned} L_{ijk} \;=\; &-\bar{\kappa}\,\{\, m_{[i}\, u_{j]}\, u_k \,-\, \frac{1}{3}\, m_{[i}\, g_{j]k} \,\} \\ &-\kappa\,\{\, \bar{m}_{[i}\, u_{j]}\, u_k \,-\, \frac{1}{3}\, \bar{m}_{[i}\, g_{j]k} \,\} \end{aligned} \qquad (73)$$

where $u^i = \frac{1}{\sqrt{2}}\,(l^i + n^j)$.

The Lanczos scalars $L_i$ $(i = 0, 1, ...7)$ in this case are found to be as follows:

$$
\begin{aligned}
L_0 &= L_{ijk}\, l^i\, m^j\, l^k = -\frac{1}{2}\,\kappa \\
L_1 &= L_{ijk}\, l^i\, m^j\, \bar{m}^k = 0 \\
L_2 &= L_{ijk}\, \bar{m}^i\, n^j\, l^k = -\frac{1}{3}\,\bar{L}_0 \\
L_3 &= L_{ijk}\, \bar{m}^i\, n^j\, \bar{m}^k = 0 \\
L_4 &= L_{ijk}\, l^i\, m^j\, m^k = 0 \\
L_5 &= L_{ijk}\, l^i\, m^j\, n^k = \frac{1}{3}\, L_0 \\
L_6 &= L_{ijk}\, \bar{m}^i\, n^j\, m^k = 0 \\
L_7 &= L_{ijk}\, \bar{m}^i\, n^j\, n^k = -\,\bar{L}_0
\end{aligned}
\tag{74}
$$

**Lemma 2.** If in a given space-time there is a field of observers $u^i$ $u^i = \frac{1}{\sqrt{2}}\,(\, l^i + n^i \,)$ which is geodetic, shear-free, expansion-free and the vorticity vector is co-variantly constant (i.e., $a_i = \theta = \sigma_{ij} = 0$ , $\omega_{i;j} = 0$) then the Lanczos potential is given by

$$
\begin{aligned}
L_{ijk} &= \frac{\sqrt{2}}{9}\,\rho\,\{\, 2(\, m_i\bar{m}_j - \bar{m}_i m_j \,)\, u_k \\
&+ (\, m_i\bar{m}_k - \bar{m}_i m_k \,)\, u_j - (\, m_j\bar{m}_k - \bar{m}_j m_k \,)\, u_i
\end{aligned}
\tag{75}
$$

where $u^i = \frac{1}{\sqrt{2}}\,(\, l^i + n^i \,)$.

The Lanczos scalars $L_i$ $(i = 0, 1, ...7)$ in this case are found to be as follows:

$$
\begin{aligned}
L_0 &= L_{ijk}\, l^i\, m^j\, l^k = 0 \\
L_1 &= L_{ijk}\, l^i\, m^j\, \bar{m}^k = \frac{1}{9}\,\rho \\
L_2 &= L_{ijk}\, \bar{m}^i\, n^j\, l^k = 0 \\
L_3 &= L_{ijk}\, \bar{m}^i\, n^j\, \bar{m}^k = 0 \\
L_4 &= L_{ijk}\, l^i\, m^j\, m^k = 0 \\
L_5 &= L_{ijk}\, l^i\, m^j\, n^k = 0 \\
L_6 &= L_{ijk}\, \bar{m}^i\, n^j\, m^k = \frac{1}{9}\,\rho \\
L_7 &= L_{ijk}\, \bar{m}^i\, n^j\, n^k = 0
\end{aligned} \tag{76}
$$

It may be noted here that the given conditions of this lemma, in terms of the spin coefficients, are equivalent to

$$
\begin{gathered}
\kappa = \nu = \sigma = \lambda = \pi = \tau = 0 \quad , \quad \alpha + \bar{\beta} = 0 \\
\epsilon + \bar{\epsilon} = \gamma + \bar{\gamma} = 0 \quad , \quad \rho = -\bar{\rho} = \mu \\
D\,\rho = D'\,\rho = \delta\,\rho = \delta'\,\rho = 0
\end{gathered} \tag{77}
$$

**Remarks:**

**1.** It has been conjectured by Lopez-Bonilla and co-workers ([7], [8], [11], [28], [45], [46]) , by considering a number of space-times, that there is some linear relationship between the Lanczos scalars and the spin coefficients. Here we have found, through equations (74) and (76), some structural link between the spin coefficients and the Lanczos scalars and thereby provide a support to the conjecture of Lopez-Bonilla and co-workers.

**2.** It is known that [26] the Gödel solution is characterized by

$$\theta = \sigma = \omega_{ij} = 0 \ , \ \omega_{i;j} = 0$$

$$\omega = \frac{1}{2}\sqrt{\omega_{ij}\,\omega^{ij}} = \frac{1}{a\sqrt{2}} = \text{constant} \qquad ([42]) \tag{78}$$

The Gödel solution is not a realistic model of the universe but it does possess a number of interesting properties. The matter in this universe does not expand but rotate. The solution also contains timelike lines, i.e., an observer can influence his past (for a detailed account of other geometrical and physical properties of the Gödel universe, see [32]).

It may be noted here that the hypothesis of lemma 2 are in fact the conditions of the Gödel solution and thus we have obtained, through equations (75) and (76), a Lanczos potential for the Gödel solution.

**3.** When the equations (76) and (77) are substituted into the NP version [15] of the Weyl-Lanczos relations, we get $\Psi_2 = \frac{2}{3}\rho^2$ which shows that the Gödel solution is of Petrov type D (this fact was missing in the paper of Novello and Velloso [54]).

## 3. Lanczos Potential and GHP - Formalism

In this section we shall translate the Weyl-Lanczos relations (16) and the Lanczos differential gauge conditions (15) into GHP formalism. These equations will then be applied to obtain the Lanczos potential for Petrov type D space-times. As an example, the Lanczos potential for the Kerr metric has been found.

By projecting the tetrad $\{l^i, n^i, m^i, \bar{m}^i\}$ on equation (16), the GHP version of the Weyl-Lanczos equation is found to be the following set of five coupled linear differential equations:

$$\Psi_0 = 2\mathcal{D}L_0 - 2\mathcal{P}L_4 - 2\tau' L_0 + 6\sigma L_1 + 2\rho L_4 + 6\kappa L_5 \qquad (79\ a)$$

$$\Psi_1 = \mathcal{P}' L_0 + \mathcal{D}L_1 - \mathcal{D}' L_4 - \mathcal{P}L_5 + (\rho' - \rho')L_0 + (3\tau - \tau')L_1$$
$$+2\sigma L_2 + (\bar{\tau} - \tau')L_4 + (\bar{\rho} - 3\rho)L_5 - 2\kappa L_6 \qquad (79\ b)$$

$$\Psi_2 = -\mathcal{P}' L_1 + \mathcal{D}L_2 - \mathcal{D}' L_5 - \mathcal{P}L_6 + \kappa' L_0 + (2\rho' - \rho')L_1 + (2\tau - \tau')L_2$$
$$+\sigma L_3 - \sigma' L_4 + (\bar{\tau} - 2\tau')L_5 + (\bar{\rho} - 2\rho)L_6 - \kappa L_7 \qquad (79\ c)$$

$$\Psi_3 = \mathcal{P}' L_2 + \mathcal{D}L_3 - \mathcal{D}' L_6 - \mathcal{P}L_7 + 2\kappa' L_1 + (3\rho' - \rho')L_2 + (\tau - \tau')L_3$$
$$-2\sigma' L_5 + (\bar{\tau} - 3\tau')L_6 + (\bar{\rho} - \rho)L_7 \qquad (79\ d)$$

$$\Psi_4 = 2\mathcal{P}' L_3 - 2\mathcal{D}' L_7 + 6\kappa' L_2 - 2\rho' L_3 - 6\sigma' L_6 + 2\tau L_7 \qquad (79\ e)$$

where the ten independent components of the Weyl spinor are given by the five complex quantities $\Psi_r, r = 0, 1, ..4$ and the sixteen independent components of the Lanczos tensor $L_{ijk}$ are given by the eight complex quantities $L_s\ ,\ \ s = 0, 1, ..7$. $L_s$ are known as the Lanczos scalars. The spin and boost types as well as the weights of these scalars are as follows:

$$L_0 = L_{ijk}\ l^i m^j l^k\ :\ \{3, 1\}\ ;\ \text{spin weight}\ =\ 1\ ,\ \text{boost weight}\ =\ 2$$

$$L_1 = L_{ijk}\ l^i m^j \bar{m}^k\ :\ \{1, 1\}\ ;\ \text{spin weight}\ =\ 0\ ,\ \text{boost weight}\ =\ 1$$

$$L_2 \;=\; L_{ijk}\,\bar{m}^i n^j l^k \;:\; \{-1,-1\}\;;\ \text{spin weight} \;=\; -1\;,\ \text{boost weight} \;=\; 0$$

$$L_3 \;=\; L_{ijk}\,\bar{m}^i n^j \bar{m}^k \;:\; \{-3,1\}\;;\ \text{spin weight} \;=\; -2\;,\ \text{boost weight} \;=\; -1$$

$$L_4 \;=\; L_{ijk}\,l^i m^j m^k \;:\; \{3,-1\}\;;\ \text{spin weight} \;=\; 2\;,\ \text{boost weight} \;=\; 1 \quad (80)$$

$$L_5 \;=\; L_{ijk}\,l^i m^j n^k \;:\; \{1,-1\}\;;\ \text{spin weight} \;=\; 1\;,\ \text{boost weight} \;=\; 0$$

$$L_6 \;=\; L_{ijk}\,\bar{m}^i n^j m^k \;:\; \{-1,-1\}\;;\ \text{spin weight} \;=\; 0\;,\ \text{boost weight} \;=\; -1$$

$$L_7 \;=\; L_{ijk}\,\bar{m}^i n^j n^k \;:\; \{-3,-1\}\;;\ \text{spin weight} \;=\; -1\;,\ \text{boost weight} \;=\; -2$$

The GHP version of the differential gauge conditions (15) is given by the following set of three equations:

$$\begin{aligned} &\mathcal{P}'L_2 + \mathcal{D}L_3 - \mathcal{D}'L_6 - \mathcal{P}L_7 + 2\kappa' L_1 - (\rho' + \bar{\rho}')L_2 + (\tau - \bar{\tau}')L_3 \\ &\qquad -2\sigma' L_5 + (\tau - 3\tau')L_6 - (\rho + \rho)L_7 = 0 \end{aligned} \qquad (81\ a)$$

$$\begin{aligned} &\mathcal{P}'L_0 + \mathcal{D}L_1 - \mathcal{D}'L_4 + \mathcal{P}L_5 - (\rho' + \bar{\rho}')L_0 + (3\tau + \bar{\tau}')L_1 \\ &\qquad -2\sigma L_2 + (\bar{\tau} + \tau')L_4 - (\bar{\rho} + 3\rho)L_5 + 2\kappa L_6 = 0 \end{aligned} \qquad (81\ b)$$

$$\mathcal{P}'L_1 - \mathcal{D}L_2 - \mathcal{D}'L_5 - \mathcal{P}L_6 + \kappa' L_0 - (2\rho' + \bar{\rho}')L_1 + (2\tau - \bar{\tau}')L_2$$

$$-\sigma L_3 - \sigma' L_4 + (\bar{\tau} + 2\tau')L_5 - (\bar{\rho} + 2\rho)L_6 + \kappa L_7 = 0 \qquad (81\ c)$$

It may be noted from the completeness relation

$$L_{ijk} \ = \ K_{ijk} \ + \ \bar{K}_{ijk} \qquad (82)$$

between the Lanczos spin tensor $L_{ijk}$ and the Lanczos scalars $L_s (s = 0, 1, ..., 7)$ where

$$K_{ijk} = L_0 U_{ij} n_k + L_1 (M_{ij} n_k - U_{ij} m_k) + L_2 (V_{ij} n_k - M_{ij} m_k) - L_3 V_{ij} m_k - L_4 U_{ij} \bar{m}_k$$

$$+ L_5 (U_{ij} l_k - M_{ij} \bar{m}_k) + L_6 (M_{ij} l_k - V_{ij} \bar{m}_k) + L_7 V_{ij} l_k \qquad (83)$$

and

$$M_{ij} \ = \ l_i n_j - l_j n_i + m_i \bar{m}_j - m_j \bar{m}_i$$

$$U_{ij} \ = \ - n_i \bar{m}_j + n_j \bar{m}_i \ , \ V_{ij} \ = \ l_i m_j - l_j m_i \qquad (84)$$

we can construct the Lanczos potential which in turn generates the Weyl tensor through equation (16).

Since the GHP formalism has proved useful in the past for studying the Petrov type D gravitational fields (cf. [1], [33]) it therefore seems worthwhile to have a study of Lanczos potential for such space-times. As the field is of Petrov type D, both $l^i$ and $n^i$ may be chosen to lie in the direction of the degenerate principal null vectors so that

$$\kappa \ = \ \sigma \ = \ \kappa' \ = \ \sigma' \ = \ 0 \qquad (85\ a)$$

$$\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0 \tag{85 b}$$

Thus for a Petrov type D space-time, from equations (85) and (16) (see also equation (114 - App.)), the Lanczos scalars are

$$L_1 = \rho \; , \; L_5 = \tau \; , \; L_i = 0 \; , \; i = 0,2,3,4,6,7 \tag{86}$$

Using the GHP field equations (23 a - I) - (23 f' - I) and the GHP Bianchi identities (27 a - I) - (27 d' - I) under the assumption (85) (see also equations (44-53 - App.) and (57-60 - App.), it can easily be shown that equation (86) is indeed a solution of the Weyl-Lanczos relations (16). Hence from equations (83) and (86), the complex Lanczos potential is

$$L_{ijk} = 2\{ \rho \, (M_{ij}n_k - U_{ij}m_k) \; + \; \tau \, (U_{ij}l_k - M_{ij}\bar{m}_k) \; \} \tag{87}$$

*which in turn generates Weyl tensor through equation (16).*

**Example:** As an illustration of the above discussions, we consider the Kerr space-time as the tetrad $\{l^i, n^i, m^i, \bar{m}^i\}$ of the GHP formalism is the natural type D space-time tetrad for the Kerr space-time. For the Kerr space-time the only non vanishing spin coefficients are $\rho, \rho', \tau$ and $\tau'$ and the non zero component of the Weyl spinor is $\Psi_2$. From equations (85), the GHP field equations (23 a - I) - (23 f'- I) and the GHP Bianchi identities (27 a - I) - (27 d' - I) (see also equations (44-53 - App.) and (57-60 - App.)) it is not hard to obtain

$$\tau = -A\rho\bar{\rho} \tag{88}$$

$$\tau' = A\rho^2 \tag{89}$$

$$\mathcal{P}\tau' = \mathcal{D}'\rho = 2\rho\tau' \tag{90}$$

$$\Psi_2 = M\rho^3 \tag{91}$$

where M is the mass parameter of the Kerr space-time and the constant of integration A satisfies equation $\mathcal{P}A = 0$ ( [10], [38]).

From equations (86), (88) and (91), the Lanczos potential for the Kerr space-time is given by

$$L_1 = \left( \frac{\Psi_2}{M} \right)^{1/3}, \; L_5 = -A \left( \frac{\Psi_2}{M} \right)^{1/3} \bar{\rho} \tag{92}$$

which shows that the Lanczos potential of the Kerr black hole is related to the mass parameter of the Kerr black hole and the Coulomb component of the gravitational field. Moreover, from equation (88), the Lanczos potential of the Kerr black hole depends only on one of the spin coefficients $\tau$ or $\rho$.

**Remark:** The Lanczos potential given by equation (86) for a Petrov type D space-time is not unique. It can be shown that

$$L_2 = -\tau', \; L_6 = -\rho', \; L_i = 0, \; i = 0,1,3,4,5,7 \tag{93}$$

is also a solution of the Weyl-Lanczos equations and consequently a Lanczos potential for the Kerr metric, where $\tau'$ and $\rho$ are related through equation (89).

## 4. Conclusion

In view of the importance of the Lanczos potential in general relativity, a detailed study of this potential has been made in this Chapter. The method of general observers has been used. The kinematical quantities, such as expansion , shear and twist etc., and the equations satisfied by them have been translated into the language of spin coefficient formalism of Newman and Penrose. The tensorial versions of the earlier results of Novello and Velloso, as described by Dolan and Kim, have been written in terms of the spin coefficients. A relationship between the Lanczos scalars and the spin coefficients is established and the conjecture of Lopez-Bonilla and co-workers has been verified. The GHP versions of the Lanczos differential gauge conditions and the Weyl-Lanczos relations have been obtained and the Lanczos potential for Petrov type D space-times has been found in terms of the spin coefficients. These results are then applied to a Kerr black hole and it is seen that the Lanczos potential of the Kerr black hole is related to the mass parameter of the Kerr black hole and the Coulomb component of the gravitational field. Moreover, the Lanczos potential of the Kerr black hole depends only on one spin coefficient.

# CHAPTER V

# Transformation Laws for the Gravitational Field Variables

## 1. Introduction

It is known that the GHP formalism deals with the quantities which 'transform properly' under those Lorentz transformations that leave invariant the two null directions, i.e., under boost in the $l-n$ plane and under rotation in the $m-\bar{m}$ plane perpendicular to these two null directions. At each point of the space-time it is always possible to introduce a complex null tetrad $Z^a_\mu$ satisfying equations (1- I) - (3 - I). These conditions are invariant under Lorentz transformations. Thus corresponding to six parameters of the homogeneous group of Lorentz transformations, we have six degrees of freedom to rotate the null tetrad. These transformation laws are given by

(a) *Null rotation about l*

$$
\begin{aligned}
l^a \longrightarrow \tilde{l}^a &= l^a \\
n^a \longrightarrow \tilde{n}^a &= n^a + A\bar{A}l^a + A\bar{m}^a + \bar{A}m^a \\
m^a \longrightarrow \tilde{m}^a &= Al^a + m^a \\
\bar{m}^a \longrightarrow \tilde{\bar{m}}^a &= \bar{A}l^a + \bar{m}^a
\end{aligned}
\qquad (1)
$$

**(b)** *Null rotation which leaves the direction of l and n unchanged but rotate m (and $\bar{m}$) in m - $\bar{m}$ plane*

$$
\begin{aligned}
l^a \longrightarrow \tilde{l}^a &= X^{-1}\, l^a \\
n^a \longrightarrow \tilde{n}^a &= X\, n^a \\
m^a \longrightarrow \tilde{m}^a &= e^{i\phi}\, m^a \\
\bar{m}^a \longrightarrow \tilde{\bar{m}}^a &= e^{-i\phi}\, \bar{m}^a
\end{aligned} \tag{2}
$$

**(c)** *Null Rotation about n*

$$
\begin{aligned}
l^a \longrightarrow \tilde{l}^a &= l^a + B\bar{m}^a + \bar{B}m^a + B\bar{B}n^a \\
n^a \longrightarrow \tilde{n}^a &= n^a \\
m^a \longrightarrow \tilde{m}^a &= m^a + Bn^a \\
\bar{m}^a \longrightarrow \tilde{\bar{m}}^a &= \bar{m}^a + \bar{B}n^a
\end{aligned} \tag{3}
$$

**(d)** *Reflection in l - n plane*

$$
\begin{aligned}
l^a \longrightarrow \tilde{l}^a &= l^a \\
n^a \longrightarrow \tilde{n}^a &= n^a \\
m^a \longrightarrow \tilde{m}^a &= \bar{m}^a \\
\bar{m}^a \longrightarrow \tilde{\bar{m}}^a &= m^a
\end{aligned} \tag{4}
$$

(e) *Reflection in m - $\bar{m}$ plane*

$$l^a \longrightarrow \tilde{l}^a = n^a$$

$$n^a \longrightarrow \tilde{n}^a = l^a$$

$$m^a \longrightarrow \tilde{m}^a = m^a$$

$$\bar{m}^a \longrightarrow \tilde{\bar{m}}^a = \bar{m}^a \qquad (5)$$

(f) *Improper complex Lorentz transformation*

$$l^a \longrightarrow \tilde{l}^a = m^a$$

$$n^a \longrightarrow \tilde{n}^a = -\bar{m}^a$$

$$m^a \longrightarrow \tilde{m}^a = -l^a$$

$$\bar{m}^a \longrightarrow \tilde{\bar{m}}^a = n^a \qquad (6)$$

These transformation laws have extensively been used in finding the symmetries of the space-times and the solutions of the Einstein's field equations. For example, Kolassis [39] has given the equations describing the geometry of a space-time admitting a $G_2$ (two dimensional group of isometries) with time-like orbits. These equations are obtained from the equations of the space-like case by means of the transformation (6); Machado Ramos and Vickers ([50], [51]) have devaloped a space-time calculus based on the null rotation (1) and recently Edgar and Ludwig ([23]) have made use of transformation (1) in finding the class of conformally flat, pure radiation metrics which are not plane waves. The null rotation freedom allowed them a choice of the spin coefficients.

Motivated by the applications of the transformation laws (1) - (6) and the importance of the GHP formalism (cf. Chapters I - IV), the present Chapter deals with the behaviour of the GHP gravitational field variables under the transformation laws (1) - (6). Not all the transformation laws for GHP variables have so far been presented in the litrature except (6), which we shall include here only for the sake of completeness. Section 2 contains all these results and a discussion of the results is given in section 3.

## 2. Transformation equations for GHP variables

The GHP formalism deals with quantities which have specific behaviour under the transformation (4 - I) and have proper spin and boost weights. These quantities are eight spin coefficients $\kappa, \sigma, \rho, \tau, \kappa', \sigma', \rho', \tau'$ ; four differential operators $\mathcal{P}, \mathcal{P}', \mathcal{D}, \mathcal{D}'$ ; five components $\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4$ of the Weyl spinor $\Psi_{ABCD}$ and nine components $\Phi_{00}, \Phi_{01}, \Phi_{02}, \Phi_{10}, \Phi_{11}, \Phi_{12}, \Phi_{20}, \Phi_{21}, \Phi_{22}$ of the trace-free Ricci spinor $\Phi_{ABC'D'}$. We shall call these quantities as the GHP gravitational field variables and find how they behave under the transformation laws (1) - (6).

(a) *Transformation under (1)*

The spin coefficients transform as follows:

$$\kappa \longrightarrow \tilde{\kappa} = \kappa$$

$$\sigma \longrightarrow \tilde{\sigma} = \sigma + A\kappa$$

$$\rho \longrightarrow \tilde{\rho} = \rho + \bar{A}\kappa$$

$$\tau \longrightarrow \tilde{\tau} = \tau + \bar{A}A\kappa + A\rho + \bar{A}\sigma$$

$$\kappa' \longrightarrow \tilde{\kappa}' = \kappa' + A\bar{A}\tau' + A\sigma' + \bar{A}\rho' - \bar{A}^2\tau - A\bar{A}^3\kappa - A\bar{A}^2\rho - \bar{A}^3\sigma$$
$$-(\mathcal{P}'+A\bar{A}\mathcal{P}+A\mathcal{D}'+\bar{A}\mathcal{D})\bar{A}+A\bar{A}^2(\mathcal{P}-D)+\bar{A}(\mathcal{P}'-D')+\bar{A}^2(\mathcal{D}-\delta)+A\bar{A}(\mathcal{D}'-\delta')$$

$$\sigma' \longrightarrow \tilde{\sigma}' = \sigma'+\bar{A}\tau'-A\bar{A}^2\bar{\kappa}-\bar{A}^3\kappa-\bar{A}^2\rho-(\bar{A}\mathcal{P}-\mathcal{D}')\bar{A}+\bar{A}^2(\mathcal{P}-D)+\bar{A}(\mathcal{D}'-\delta')$$

$$\rho' \longrightarrow \tilde{\rho}' = \rho' + A\tau' - A^2\bar{A}\kappa - \bar{A}^2\sigma - (A\mathcal{P} + \mathcal{D})\bar{A} + A\bar{A}(\mathcal{P} - D) + \bar{A}(\mathcal{D} - \delta)$$

$$\tau' \longrightarrow \tilde{\tau}' = \tau' - \bar{A}^2\kappa - \mathcal{P}\bar{A} - \bar{A}(\mathcal{P} - D)$$

The intrinsic derivative operators transform as follows:

$$D \longrightarrow \tilde{D} = D$$

$$D' \longrightarrow \tilde{D}' = D' + A\bar{A}D + A\delta' + \bar{A}\delta$$

$$\delta \longrightarrow \tilde{\delta} = \delta + AD$$

$$\delta' \longrightarrow \tilde{\delta} = \delta' + \bar{A}D$$

The GHP derivatives transform as under:

$$\mathcal{P} \longrightarrow \tilde{\mathcal{P}} = \mathcal{P} - p\bar{A}\kappa - qA\bar{\kappa}$$

$$\mathcal{P}' \longrightarrow \tilde{\mathcal{P}}' = \mathcal{P}' + A\bar{A}\mathcal{P} - \bar{A}\mathcal{D} - A\mathcal{D}' + p(\bar{A}\tau + A\bar{A}^2\kappa + A\bar{A}\rho + \bar{A}^2\sigma)$$
$$+q(A\bar{\tau} + \bar{A}A^2\bar{\kappa} + A\bar{A}\bar{\rho} + A^2\bar{\sigma})$$

$$\mathcal{D} \longrightarrow \tilde{\mathcal{D}} = \mathcal{D} + A\mathcal{P} - p(A\bar{A}\kappa + \bar{A}\sigma) - q(A^2\bar{\kappa} + A\bar{\rho})$$

$$\mathcal{D}' \longrightarrow \tilde{\mathcal{D}}' = \mathcal{D}' + \bar{A}\mathcal{P} - p(\bar{A}^2\kappa + \bar{A}\rho) - q(A\bar{A}\bar{\kappa} + A\bar{\sigma})$$

The components of Weyl conformal and trace-free Ricci tensors transform as follows:

$$\Psi_0 \longrightarrow \tilde{\Psi}_0 = \Psi_0$$

$$\Psi_1 \longrightarrow \tilde{\Psi}_1 = \Psi_1 + \bar{A}\Psi_0$$

$$\Psi_2 \longrightarrow \tilde{\Psi}_2 = \Psi_2 + 2\bar{A}\Psi_1 + \bar{A}^2\Psi_0$$

$$\Psi_3 \longrightarrow \tilde{\Psi}_3 = \Psi_3 + 3\bar{A}\Psi_2 + 3\bar{A}^2\Psi_1 + \bar{A}^3\Psi_0$$

$$\Psi_4 \longrightarrow \tilde{\Psi}_4 = \Psi_4 + 4\bar{A}\Psi_3 + 6\bar{A}^2\Psi_2 + 4\bar{A}^3\Psi_1 + \bar{A}^4\Psi_0$$

$$\Phi_{00} \longrightarrow \tilde{\Phi}_{00} = \Phi_{00}$$

$$\Phi_{01} \longrightarrow \tilde{\Phi}_{01} = \Phi_{01} + A\Phi_{00}$$

$$\Phi_{02} \longrightarrow \tilde{\Phi}_{02} = \Phi_{02} + 2A\Phi_{01} + A^2\Phi_{00}$$

$$\Phi_{11} \longrightarrow \tilde{\Phi}_{11} = \Phi_{11} + A\Phi_{10} + \bar{A}\Phi_{01} + A\bar{A}\Phi_{00}$$

$$\Phi_{12} \longrightarrow \tilde{\Phi}_{12} = \Phi_{12} + A^2\Phi_{10} + 2A\Phi_{11} + \bar{A}\Phi_{02} + 2A\bar{A}\Phi_{01} + \bar{A}A^2\Phi_{00}$$

$$\Phi_{22} \longrightarrow \tilde{\Phi}_{22} = \Phi_{22} + 2A\Phi_{21} + A^2\Phi_{20} + 2\bar{A}\Phi_{12} + 4A\bar{A}\Phi_{11} + 2\bar{A}A^2\Phi_{10}$$
$$+\bar{A}^2\Phi_{02} + 2A\bar{A}^2\Phi_{01} + A^2\bar{A}^2\Phi_{00}$$

(b) *Transformation under (2)*

The spin coefficients, intrinsic derivatives, GHP derivatives, the components of Weyl and Ricci tensors transform as follows:

$$\kappa \longrightarrow \tilde{\kappa} = e^{i\phi}\kappa$$

$$\sigma \longrightarrow \tilde{\sigma} = Xe^{2i\phi}\sigma$$

$$\rho \longrightarrow \tilde{\rho} = X\rho$$

$$\tau \longrightarrow \tilde{\tau} = X^2e^{i\phi}\tau$$

$$\kappa' \longrightarrow \tilde{\kappa}' = -e^{i\phi}\kappa'$$

$$\sigma' \longrightarrow \tilde{\sigma}' = -X^{-1}e^{-2i\phi}\sigma'$$

$$\rho' \longrightarrow \tilde{\rho}' = -X^{-1}\rho'$$

$$\tau' \longrightarrow \tilde{\tau}' = -X^{-2}e^{-i\phi}\tau'$$

$$D \longrightarrow \tilde{D} = X^{-1} D$$

$$D' \longrightarrow \tilde{D}' = X D'$$

$$\delta \longrightarrow \tilde{\delta} = e^{i\phi} \delta$$

$$\delta' \longrightarrow \tilde{\delta}' = e^{-i\phi} \delta'$$

$$\mathcal{P} \longrightarrow \tilde{\mathcal{P}} = \mathcal{P} - \frac{1}{2} p \left(-XDX + iD\phi\right) - \frac{1}{2} q X^{-1} \left(-X^{-2} D\bar{X} + i\bar{X} D\phi\right)$$

$$\mathcal{P}' \longrightarrow \tilde{\mathcal{P}}' = \mathcal{P}' + \frac{1}{2} p \left(X^{-1} DX + iD\phi\right) + \frac{1}{2} q \left(X^{-1} D\bar{X} + i\bar{X} X^{-1} D\phi\right)$$

$$\mathcal{D} \longrightarrow \tilde{\mathcal{D}} = \mathcal{D} - \frac{1}{2} p \left(X\delta X + \delta\phi\right) + \frac{1}{2} q \left(\bar{X} e^{-2i\phi} \delta\bar{X} - i\delta\phi\right)$$

$$\mathcal{D}' \longrightarrow \tilde{\mathcal{D}}' = \mathcal{D}' + \frac{1}{2} p \left(X\delta' X - i\delta'\phi\right) - \frac{1}{2} q \left(i\bar{\delta}\phi - \bar{X}\bar{\delta}\bar{X}\right)$$

$$\Psi_0 \longrightarrow \tilde{\Psi}_0 = X^{-2} e^{2i\phi} \Psi_0$$

$$\Psi_1 \longrightarrow \tilde{\Psi}_1 = X^{-1} e^{i\phi} \Psi_1$$

$$\Psi_2 \longrightarrow \tilde{\Psi}_2 = \Psi_2$$

$$\Psi_3 \longrightarrow \tilde{\Psi}_3 = X e^{-i\phi} \Psi_3$$

$$\Psi_4 \longrightarrow \tilde{\Psi}_4 = X^2 e^{-2i\phi}\Psi_4$$

$$\Phi_{00} \longrightarrow \tilde{\Phi}_{00} = X^{-2}\Phi_{00}$$

$$\Phi_{01} \longrightarrow \tilde{\Phi}_{01} = X^{-1}e^{i\phi}\Phi_{01}$$

$$\Phi_{02} \longrightarrow \tilde{\Phi}_{02} = e^{2i\phi}\Phi_{02}$$

$$\Phi_{11} \longrightarrow \tilde{\Phi}_{11} = \Phi_{11}$$

$$\Phi_{12} \longrightarrow \tilde{\Phi}_{12} = Xe^{i\phi}\Phi_{12}$$

$$\Phi_{22} \longrightarrow \tilde{\Phi}_{22} = X^2\Phi_{22}$$

(c) *Transformation under (3)*

The spin coefficients, intrinsic derivatives, GHP derivatives, the components of Weyl and Ricci tensors transform as follows:

$$\kappa \longrightarrow \tilde{\kappa} = \kappa + B\rho + \bar{B}\sigma + B\bar{B}\tau - B^3\bar{B}\kappa' - B^2\bar{B}\rho' - B^3\sigma' - B^2\tau'$$
$$-(\mathcal{P}-B\bar{B}\mathcal{P}'-B\mathcal{D}'-\bar{B}\mathcal{D})B+B(\mathcal{P}-D)+B^2\bar{B}(\mathcal{P}'-D')+B\bar{B}(\mathcal{D}-\delta)+B^2(\mathcal{D}'-\delta')$$

$$\sigma \longrightarrow \tilde{\sigma} = \sigma + B\tau - B^2\rho' - B^3\kappa' - \mathcal{D}B - B\mathcal{P}'B - B(\mathcal{D}-\delta) - B^2(\mathcal{P}'-D')$$

$$\rho \longrightarrow \tilde{\rho} = \rho + \bar{B}\tau - B^2\sigma' + B^2\bar{B}\kappa' - \mathcal{D}'B - \bar{B}\mathcal{P}'B + B(\mathcal{D}'-\delta') + B\bar{B}(\mathcal{P}'-D')$$

$$\tau \longrightarrow \tilde{\tau} = \tau - B^2\kappa' - \mathcal{P}'B - \bar{B}(\mathcal{P}' - D')$$

$$\kappa' \longrightarrow \tilde{\kappa}' = \kappa'$$

$$\sigma' \longrightarrow \tilde{\sigma}' = \sigma' + \bar{B}\kappa'$$

$$\rho' \longrightarrow \tilde{\rho}' = \rho' + B\kappa'$$

$$\tau' \longrightarrow \tilde{\tau}' = \tau' + B\sigma' + \bar{B}\rho' + B\bar{B}\kappa'$$

$$D \longrightarrow \tilde{D} = D + B\delta' + \bar{B}\delta + B\bar{B}D'$$

$$D' \longrightarrow \tilde{D}' = D'$$

$$\delta \longrightarrow \tilde{\delta} = \delta + BD'$$

$$\delta' \longrightarrow \tilde{\delta}' = \delta' + \bar{B}D'$$

$$\mathcal{P} \longrightarrow \tilde{\mathcal{P}} = \mathcal{P} + B\bar{B}\mathcal{P}' + \bar{B}\mathcal{D} + B\mathcal{D}'$$
$$+p(B\tau' + B^2\sigma' + B\bar{B}\rho' + B^2\bar{B}\kappa') + q(\bar{B}\bar{\tau}' + \bar{B}^2\bar{\sigma}' + B\bar{B}\bar{\rho}' + \bar{B}^2B\bar{\kappa}')$$

$$\mathcal{P}' \longrightarrow \tilde{\mathcal{P}}' = \mathcal{P}' + pB\kappa' + q\bar{B}\bar{\kappa}'$$

$$\mathcal{D} \longrightarrow \tilde{\mathcal{D}} = \mathcal{D} + B\mathcal{P}' + p(B\rho' + B^2\kappa') + q(\bar{B}\bar{\sigma}' + B\bar{B}\bar{\kappa}')$$

$$\mathcal{D}' \longrightarrow \tilde{\mathcal{D}}' = \mathcal{D}' + \bar{B}\mathcal{P}' + p(B\sigma' + B\bar{B}\kappa') + q(\bar{B}\bar{\rho}' + \bar{B}^2\bar{\kappa}')$$

$$\Psi_0 \longrightarrow \tilde{\Psi}_0 = \Psi_0 + 4B\Psi_1 + 6B^2\Psi_2 + 4B^3\Psi_3 + B^4\Psi_4$$

$$\Psi_1 \longrightarrow \tilde{\Psi}_1 = \Psi_1 + 3B\Psi_2 + 3B^2\Psi_3 + B^3\Psi_4$$

$$\Psi_2 \longrightarrow \tilde{\Psi}_2 = \Psi_2 + 2B\Psi_3 + B^2\Psi_4$$

$$\Psi_3 \longrightarrow \tilde{\Psi}_3 = \Psi_3 + B\Psi_4$$

$$\Psi_4 \longrightarrow \tilde{\Psi}_4 = \Psi_4$$

$$\begin{aligned}\Phi_{00} \longrightarrow \tilde{\Phi}_{00} = \Phi_{00} &+ 2\bar{B}\Phi_{01} + 2B\Phi_{10} + 4\bar{B}\Phi_{11} + \bar{B}^2\Phi_{02} \\ &+ B^2\Phi_{20} + 2\bar{B}B^2\Phi_{21} + 2\bar{B}^2\Phi_{12} + \bar{B}^2B^2\Phi_{22}\end{aligned}$$

$$\Phi_{10} \longrightarrow \tilde{\Phi}_{10} = \Phi_{10} + 2\bar{B}\Phi_{11} + B\Phi_{20} + 2B\bar{B}\Phi_{21} + \bar{B}^2\Phi_{12} + \bar{B}^2B\Phi_{22}$$

$$\Phi_{11} \longrightarrow \tilde{\Phi}_{11} = \Phi_{11} + \bar{B}\Phi_{12} + B\Phi_{21} + B\bar{B}\Phi_{22}$$

$$\Phi_{20} \longrightarrow \tilde{\Phi}_{20} = \Phi_{20} + 2\bar{B}\Phi_{21} + \bar{B}^2\Phi_{22}$$

$$\Phi_{21} \longrightarrow \tilde{\Phi}_{21} = \Phi_{21} + \bar{B}\Phi_{22}$$

$$\Phi_{22} \longrightarrow \tilde{\Phi}_{22} = \Phi_{22}$$

The transformation law discussed above can lead to the Petrov classification of the gravitational fields as follows:

We know that the Weyl conformal tensor is completely specified by the five complex scalars $\Psi_i$, $i = 0, 1, ..4$, with respect to the null tetrad $\{l^a, n^a, m^a, \bar{m}^a\}$. However, they can be transformed with the help of the three types of rotations mentioned above (cf. equations (1) - (3)). The possible types of rotations which make one or more of the complex scalars to vanish is the key for the algebraic classification of the gravitational fields. Let $\Psi_4 \neq 0$ then by a rotation of type given by equation (3) we get

$$\tilde{\Psi}_0 = \Psi_0 + 4B\Psi_1 + 6B^2\Psi_2 + 4B^3\Psi_3 + B^4\Psi_4$$

The scalar $\tilde{\Psi}_0$ will vanish if $B$ is the root of the biquadratic

$$\Psi_4 B^4 + 4\Psi_3 B^3 + 6\Psi_2 B^2 + 4\Psi_1 B + \Psi_0 = 0 \tag{7}$$

Simultaneously, the null direction $l^a$ is transformed to $l^a = l^a + \bar{B}m^a + B\bar{m}^a + B\bar{B}n^a$ and corresponding to each $B$ there is a new null direction. These new four null directions are called principal null directions of the Weyl conformal tensor. If the four roots, say $B_1, B_2, B_3$ and $B_4$ are all distinct, then the corresponding null directions are also distinct and the Weyl conformal tensor is said to be algebraically general. In the case when two or more roots are equal, then the corresponding principal null directions will coincide and the Weyl conformal tensor is called algebraically special. This coincidence (in different ways) and distinctness of the roots lead to the different types of the Petrov's classification of the gravitational fields. For example, when all four roots are distinct we have Petrov type I gravitational field and if $B_1 = B_2 \neq B_3 \neq B_4$ we then have Petrov type II gravitational field and so on (see also [12]).

(d) *Transformation under (4)*

The spin coefficients, intrinsic derivatives, GHP derivatives, the components of Weyl and Ricci tensors transform as follows:

$$\kappa \longrightarrow \tilde{\kappa} = \bar{\kappa} \quad , \quad \sigma \longrightarrow \tilde{\sigma} = \bar{\sigma}$$

$$\rho \longrightarrow \tilde{\rho} = \bar{\rho} \quad , \quad \tau \longrightarrow \tilde{\tau} = \bar{\tau}$$

$$\kappa' \longrightarrow \tilde{\kappa}' = \bar{\kappa}' \quad , \quad \sigma' \longrightarrow \tilde{\sigma}' = \bar{\sigma}'$$

$$\rho' \longrightarrow \tilde{\rho}' = \bar{\rho}' \quad , \quad \tau' \longrightarrow \tilde{\tau}' = \bar{\tau}'$$

$$D \longrightarrow \tilde{D} \ = D \quad , \quad D' \longrightarrow \tilde{D}' \ = D'$$

$$\delta \longrightarrow \tilde{\delta} \ = \bar{\delta} \quad , \quad \delta' \longrightarrow \tilde{\delta}' \ = \bar{\delta}'$$

$$\mathcal{P} \longrightarrow \tilde{\mathcal{P}} = \bar{\mathcal{P}} = \mathcal{P} \quad , \quad \mathcal{P}' \longrightarrow \tilde{\mathcal{P}}' = \bar{\mathcal{P}}' = \mathcal{P}'$$

$$\mathcal{D} \longrightarrow \tilde{\mathcal{D}} = \bar{\mathcal{D}} = \mathcal{D}' \quad \mathcal{D}' \longrightarrow \tilde{\mathcal{D}}' = \bar{\mathcal{D}}' = \mathcal{D}$$

$$\Psi_0 \longrightarrow \tilde{\Psi}_0 = \bar{\Psi}_0 \quad , \quad \Psi_1 \longrightarrow \tilde{\Psi}_1 = \bar{\Psi}_1$$

$$\Psi_2 \longrightarrow \tilde{\Psi}_2 = \bar{\Psi}_2$$

$$\Psi_3 \longrightarrow \tilde{\Psi}_3 = \bar{\Psi}_3 \quad , \quad \Psi_4 \longrightarrow \tilde{\Psi}_4 = \bar{\Psi}_4$$

$$\Phi_{00} \longrightarrow \tilde{\Phi}_{00} = \bar{\Phi}_{00} = \Phi_{00}$$

$$\Phi_{01} \longrightarrow \tilde{\Phi}_{01} = \bar{\Phi}_{01} = \Phi_{10}$$

$$\Phi_{02} \longrightarrow \tilde{\Phi}_{02} = \bar{\Phi}_{02} = \Phi_{20}$$

$$\Phi_{10} \longrightarrow \tilde{\Phi}_{10} = \bar{\Phi}_{10} = \Phi_{01}$$

$$\Phi_{11} \longrightarrow \tilde{\Phi}_{11} = \bar{\Phi}_{11} = \Phi_{11}$$

$$\Phi_{12} \longrightarrow \tilde{\Phi}_{12} = \bar{\Phi}_{12} = \Phi_{21}$$

$$\Phi_{20} \longrightarrow \tilde{\Phi}_{20} = \bar{\Phi}_{20} = \Phi_{02}$$

$$\Phi_{21} \longrightarrow \tilde{\Phi}_{21} = \bar{\Phi}_{21} = \Phi_{12}$$

$$\Phi_{22} \longrightarrow \tilde{\Phi}_{22} = \bar{\Phi}_{22} = \Phi_{22}$$

It may be noted from here that all the GHP field variables transform into their complex conjugates.

(e) *Transformation under (5)*

The spin coefficients transform as follows:

$$\kappa \longrightarrow \tilde{\kappa} = \bar{\kappa}' \quad , \quad \sigma \longrightarrow \tilde{\sigma} = \bar{\sigma}'$$

$$\rho \longrightarrow \tilde{\rho} = \bar{\rho}' \quad , \quad \tau \longrightarrow \tilde{\tau} = \bar{\tau}'$$

$$\kappa' \longrightarrow \tilde{\kappa}' = \bar{\kappa} \quad , \quad \sigma' \longrightarrow \tilde{\sigma}' = \bar{\sigma}$$

$$\rho' \longrightarrow \tilde{\rho}' = \bar{\rho} \quad , \quad \tau' \longrightarrow \tilde{\tau}' = \bar{\tau}$$

The intrinsic derivatives transform as follows:

$$D \longrightarrow \tilde{D} = D' \ , \ \ D' \longrightarrow \tilde{D}' = D$$

$$\delta \longrightarrow \tilde{\delta} = \delta \ , \ \ \delta' \longrightarrow \tilde{\delta}' = \delta'$$

The GHP derivatives transform as follows:

$$\mathcal{P} \longrightarrow \tilde{\mathcal{P}} = -\bar{\mathcal{P}}' = -\mathcal{P}'$$

$$\mathcal{P}' \longrightarrow \tilde{\mathcal{P}}' = -\bar{\mathcal{P}} = -\mathcal{P}$$

$$\mathcal{D} \longrightarrow \tilde{\mathcal{D}} = -\bar{\mathcal{D}}' = -\mathcal{D}$$

$$\mathcal{D}' \longrightarrow \tilde{\mathcal{D}}' = -\bar{\mathcal{D}} = -\mathcal{D}'$$

The tetrad components of the Weyl tensor transform as follows:

$$\Psi_0 \longrightarrow \tilde{\Psi}_0 = \bar{\Psi}_4$$

$$\Psi_1 \longrightarrow \tilde{\Psi}_1 = \bar{\Psi}_3$$

$$\Psi_2 \longrightarrow \tilde{\Psi}_2 = \bar{\Psi}_2$$

$$\Psi_3 \longrightarrow \tilde{\Psi}_3 = \bar{\Psi}_1$$

$$\Psi_4 \longrightarrow \tilde{\Psi}_4 = \bar{\Psi}_0$$

The tetrad components of the trace-free Ricci tensor transform as follows:

$$\Phi_{00} \longrightarrow \tilde{\Phi}_{00} = \Phi_{22} = \bar{\Phi}_{22}$$

$$\Phi_{01} \longrightarrow \tilde{\Phi}_{01} = \Phi_{12} = \bar{\Phi}_{21}$$

$$\Phi_{02} \longrightarrow \tilde{\Phi}_{02} = \Phi_{02} = \bar{\Phi}_{20}$$

$$\Phi_{10} \longrightarrow \tilde{\Phi}_{10} = \Phi_{21} = \bar{\Phi}_{12}$$

$$\Phi_{11} \longrightarrow \tilde{\Phi}_{11} = \Phi_{11} = \bar{\Phi}_{11}$$

$$\Phi_{12} \longrightarrow \tilde{\Phi}_{12} = \Phi_{01} = \bar{\Phi}_{10}$$

$$\Phi_{20} \longrightarrow \tilde{\Phi}_{20} = \Phi_{20} = \bar{\Phi}_{02}$$

$$\Phi_{21} \longrightarrow \tilde{\Phi}_{21} = \Phi_{10} = \bar{\Phi}_{01}$$

$$\Phi_{22} \longrightarrow \tilde{\Phi}_{22} = \Phi_{00} = \bar{\Phi}_{00}$$

(f) *Transformation under (6)*

The spin coefficients, GHP derivatives, the components of Weyl and Ricci tensors transform as follows:

$$\kappa \longrightarrow \tilde{\kappa} = \sigma \quad , \quad \sigma \longrightarrow \tilde{\sigma} = -\kappa$$

$$\rho \longrightarrow \tilde{\rho} = \tau \quad , \quad \tau \longrightarrow \tilde{\tau} = -\rho$$

$$\kappa' \longrightarrow \tilde{\kappa}' = -\sigma' \quad , \quad \sigma' \longrightarrow \tilde{\sigma}' = \kappa'$$

$$\rho' \longrightarrow \tilde{\rho}' = -\tau' \quad , \quad \tau' \longrightarrow \tilde{\tau}' = \rho'$$

$$\bar{\kappa} \longrightarrow \tilde{\bar{\kappa}} = -\bar{\sigma}' \quad , \quad \bar{\sigma} \longrightarrow \tilde{\bar{\sigma}} = -\bar{\kappa}'$$

$$\bar{\rho} \longrightarrow \tilde{\bar{\rho}} = \bar{\tau}' \quad , \quad \bar{\tau} \longrightarrow \tilde{\bar{\tau}} = \bar{\rho}'$$

$$\bar{\kappa}' \longrightarrow \tilde{\bar{\kappa}}' = \bar{\sigma} \quad , \quad \bar{\sigma}' \longrightarrow \tilde{\bar{\sigma}} = \bar{\kappa}$$

$$\bar{\rho}' \longrightarrow \tilde{\bar{\rho}}' = -\bar{\tau} \quad , \quad \bar{\tau}' \longrightarrow \tilde{\bar{\tau}}' = -\bar{\rho}$$

$$\mathcal{P} \longrightarrow \tilde{\mathcal{P}} = \mathcal{D} \quad , \quad \mathcal{P}' \longrightarrow \tilde{\mathcal{P}}' = -\mathcal{D}'$$

$$\mathcal{D} \longrightarrow \tilde{\mathcal{D}} = -\mathcal{P} \quad , \quad \mathcal{D}' \longrightarrow \tilde{\mathcal{D}}' = \mathcal{P}'$$

$$\Psi_0 \longrightarrow \tilde{\Psi}_0 = \Psi_0 \quad , \quad \Psi_1 \longrightarrow \tilde{\Psi}_1 = \Psi_1$$

$$\Psi_2 \longrightarrow \tilde{\Psi}_2 = \Psi_2$$

$$\Psi_3 \longrightarrow \tilde{\Psi}_3 = \Psi_3 \quad , \quad \Psi_4 \longrightarrow \tilde{\Psi}_4 = \Psi_4$$

$$\bar{\Psi}_0 \longrightarrow \tilde{\bar{\Psi}}_0 = \bar{\Psi}_4 \quad , \quad \bar{\Psi}_1 \longrightarrow \tilde{\bar{\Psi}}_1 = -\bar{\Psi}_3$$

$$\bar{\Psi}_2 \longrightarrow \bar{\bar{\Psi}}_2 = \bar{\Psi}_2$$

$$\bar{\Psi}_3 \longrightarrow \bar{\bar{\Psi}}_3 = -\bar{\Psi}_1 \quad , \quad \bar{\Psi}_4 \longrightarrow \bar{\bar{\Psi}}_4 = \bar{\Psi}_0$$

$$\Phi_{00} \longrightarrow \tilde{\tilde{\Phi}}_{00} = \Phi_{02} \quad , \quad \Phi_{01} \longrightarrow \tilde{\tilde{\Phi}}_{01} = -\Phi_{01}$$

$$\Phi_{02} \longrightarrow \tilde{\tilde{\Phi}}_{02} = \Phi_{00} \quad , \quad \Phi_{10} \longrightarrow \tilde{\tilde{\Phi}}_{10} = \Phi_{12}$$

$$\Phi_{11} \longrightarrow \tilde{\tilde{\Phi}}_{11} = -\Phi_{11} \quad , \quad \Phi_{12} \longrightarrow \tilde{\tilde{\Phi}}_{12} = \Phi_{10}$$

$$\Phi_{20} \longrightarrow \tilde{\tilde{\Phi}}_{20} = \Phi_{22} \quad , \quad \Phi_{21} \longrightarrow \tilde{\tilde{\Phi}}_{21} = -\Phi_{21} \quad , \quad \Phi_{22} \longrightarrow \tilde{\tilde{\Phi}}_{22} = \Phi_{20}$$

It may be noted that under the transformation (6), the GHP field equations (23 - I) are permuted among themselves and so as the full Bianchi identities (25 - I) and the GHP commutators (24 - I). The simplification in generating the equations can be made by the transformation law given by equation (6) alongwith the prime operation (14 - I) and (15 - I) and thus it provides a check of equations which are obtained by other means.

## 3. Conclusion

The behaviour of the GHP gravitational field variables $\kappa, \sigma, \rho, \tau, \kappa', \sigma', \rho', \tau'; \mathcal{P}, \mathcal{P}', \mathcal{D}, \mathcal{D}'; \Psi_i (i = 0, 1, 2, 3, 4), \Phi_{00}, \Phi_{01}, .., \Phi_{22}$, have been investigated under the transformation laws given by equations (1) - (6). It is seen that the transformation law (3) leads to the classifiaction of the gravitational field in a natural way, while the transformation law (6) provides a check of GHP equations (cf. equations (23 - I) - (27 - I) and equations appearing in section 1 of Appendix, i.e., GHP field equations, commutator relations and Bianchi identities) which are obtained by other means. It may also be noted that under transformation (4) all the GHP variables transform into their complex conjugates. The expressions derived in this Chapter may be used in studying the electromagnetic and gravitational perturbations of the Schwartzchild and Kerr balck holes (see also [12]).

# APPENDIX

We shall mention here some of the results that have been used in the simplifications of a number of equations occuring in different Chapters of the thesis.

## 1. GHP Equations and Petrov Classification

In this section, we shall write down the GHP field equations, GHP commutator relations and GHP vacuum Bianchi identities for different Petrov types.

**(a) Petrov Type I :** For this type a null tetrad can be chosen such that Newman-Penrose components of the Weyl tensor in that tetrad are $\Psi_1 = \Psi_3 = 0$ , $\Psi_r \neq 0$ , $r = 0, 2, 4$.

*GHP Field Equations*

$$\mathcal{P}\rho - \mathcal{D}\kappa = \rho^2 + \sigma\bar{\sigma} - \bar{\kappa}\tau - \tau'\kappa \tag{1}$$

$$\mathcal{P}'\rho' - \mathcal{D}'\kappa' = \rho'^2 + \sigma'\bar{\sigma}' - \bar{\kappa}'\tau' - \tau\kappa' \tag{2}$$

$$\mathcal{P}\sigma - \mathcal{D}\kappa = (\rho + \bar{\rho})\sigma - (\tau + \bar{\tau}')\kappa + \Psi_0 \tag{3}$$

$$\mathcal{P}'\sigma' - \mathcal{D}'\kappa' = (\rho' + \bar{\rho}')\sigma' - (\tau' + \bar{\tau})\kappa' + \Psi_4 \tag{4}$$

$$\mathcal{P}\tau - \mathcal{P}'\kappa = (\tau - \bar{\tau}')\rho + (\bar{\tau} - \tau')\sigma \tag{5}$$

$$\mathcal{P}'\tau' - \mathcal{P}\kappa' = (\tau' - \bar{\tau})\rho' + (\bar{\tau}' - \tau)\sigma' \tag{6}$$

$$\mathcal{D}\rho - \mathcal{D}'\sigma = (\rho - \bar{\rho})\tau + (\rho' - \bar{\rho}')\kappa \tag{7}$$

$$\mathcal{D}'\rho' - \mathcal{D}\sigma' = (\rho' - \bar{\rho}')\tau' + (\rho - \bar{\rho})\kappa' \tag{8}$$

$$\mathcal{D}\tau - \mathcal{P}'\sigma = -\rho'\sigma - \bar{\sigma}'\rho + \tau^2 + \kappa\bar{\kappa}' \tag{9}$$

$$\mathcal{D}'\tau' - \mathcal{P}\sigma' = -\rho\sigma' - \bar{\sigma}\rho' + \tau'^2 + \kappa'\bar{\kappa} \tag{10}$$

$$\mathcal{P}'\rho - \mathcal{D}'\tau = -\rho\bar{\rho}' - \tau\bar{\tau} - \kappa\kappa' - \Psi_2 \tag{11}$$

$$\mathcal{P}\rho' - \mathcal{D}\tau' = -\rho'\bar{\rho} - \tau'\bar{\tau}' - \kappa'\kappa - \Psi_2 \tag{12}$$

*GHP Commutator Relations*

$$\begin{aligned}[\mathcal{P}, \mathcal{P}']\eta = \{&(\bar{\tau} - \tau')\mathcal{D} + (\tau - \bar{\tau}')\mathcal{D}' - p(\kappa\kappa' - \tau\tau' + \Psi_2) \\ &-q(\bar{\kappa}\bar{\kappa}' - \bar{\tau}\bar{\tau}' + \Psi_2)\}\eta \end{aligned} \tag{13}$$

$$[\mathcal{P}, \mathcal{D}]\eta = \{\bar{\rho}\mathcal{D} + \sigma\mathcal{D}' - \bar{\tau}'\mathcal{P} - \kappa\mathcal{P}' - p(\rho'\kappa - \tau'\sigma) - q(\bar{\sigma}'\bar{\kappa} - \bar{\rho}\bar{\tau}')\}\eta \tag{14}$$

$$\begin{aligned}[\mathcal{D}, \mathcal{D}']\eta = \{&(\bar{\rho}' - \rho')\mathcal{P} + (\rho - \bar{\rho}')\mathcal{P}' - p(\rho\rho' - \sigma\sigma' + \Psi_2) \\ &-q(\bar{\rho}\bar{\rho}' - \bar{\sigma}\bar{\sigma}' + \Psi_2)\}\eta \end{aligned} \tag{15}$$

together with the remaining commutator relations obtained by applying prime, complex conjugation, and both to equation (14). Care must be taken when applying primes and bars to these equations, as $\eta', \bar{\eta}$ and $\bar{\eta}'$ have types different to that of $\eta$. Under the prime, $p$ becomes $-p$ and $q$ becomes $-q$; under bar, $p$ becomes $q$ and $q$ becomes $p$; under both bar and prime, $p$ becomes $-q$ and $q$ becomes $-p$.

*GHP Vacuum Bianchi Identities*

$$\mathcal{D}'\Psi_0 = \tau'\Psi_0 + 3\kappa\Psi_2 \tag{16}$$

$$\mathcal{D}\Psi_4 = \tau\Psi_4 + 3\kappa'\Psi_2 \tag{17}$$

$$\mathcal{P}\Psi_2 = \sigma'\Psi_0 + 3\rho\Psi_2 \tag{18}$$

$$\mathcal{P}'\Psi_2 = \sigma\Psi_4 + 3\rho'\Psi_2 \tag{19}$$

$$\mathcal{P}\Psi_4 = 3\sigma'\Psi_2 + \rho\Psi_4 \tag{20}$$

$$\mathcal{P}'\Psi_0 = 3\sigma\Psi_2 + \rho'\Psi_0 \tag{21}$$

$$\mathcal{D}'\Psi_2 = 3\tau'\Psi_2 + \kappa\Psi_4 \tag{22}$$

$$\mathcal{D}\Psi_2 = 3\tau\Psi_2 + \kappa'\Psi_0 \tag{23}$$

**(b) Petrov Type II :** In this type, a null tetrad can be chosen such that the components of the Weyl scalars are $\Psi_0 = \Psi_1 = \Psi_3 = 0$ , $\Psi_r \neq 0$ , $r = 2, 4$ . Here, $\kappa = \sigma = 0$.

*GHP Field Equations*

$$\mathcal{P}\rho = \rho^2 \tag{24}$$

$$\mathcal{P}'\rho' - \mathcal{D}'\kappa' = \rho'^2 + \sigma'\bar{\sigma}' - \bar{\kappa}'\tau' - \tau\kappa' \tag{25}$$

$$\mathcal{P}'\sigma' - \mathcal{D}'\kappa' = (\rho' + \bar{\rho}')\sigma' - (\tau' + \bar{\tau})\kappa' + \Psi_4 \tag{26}$$

$$\mathcal{P}\tau = (\tau - \bar{\tau}')\rho \tag{27}$$

$$\mathcal{P}'\tau' - \mathcal{P}\kappa' = (\tau' - \bar{\tau})\rho' + (\bar{\tau}' - \tau)\sigma' \tag{28}$$

$$\mathcal{D}\rho = (\rho - \bar{\rho})\tau \tag{29}$$

$$\mathcal{D}'\rho' - \mathcal{D}\sigma' = (\rho' - \bar{\rho}')\tau' + (\rho - \bar{\rho})\kappa' \tag{30}$$

$$\mathcal{D}\tau = -\bar{\sigma}'\rho + \tau^2 \tag{31}$$

$$\mathcal{D}'\tau' - \mathcal{P}\sigma' = -\rho\sigma' + \tau'^2 \tag{32}$$

$$\mathcal{P}'\rho - \mathcal{D}'\tau = -\rho\bar{\rho}' - \tau\bar{\tau} - \Psi_2 \tag{33}$$

$$\mathcal{P}\rho' - \mathcal{D}\tau' = -\rho'\bar{\rho} - \tau'\bar{\tau}' - \Psi_2 \tag{34}$$

*GHP Commutator Relations*

$$[\mathcal{P}, \mathcal{P}']\eta = \{(\bar{\tau} - \tau')\mathcal{D} + (\tau - \bar{\tau}')\mathcal{D}' - p(-\tau\tau' + \Psi_2) - q(-\bar{\tau}\bar{\tau}' + \Psi_2)\}\eta \tag{35}$$

$$[\mathcal{P}, \mathcal{D}]\eta = \{\bar{\rho}\mathcal{D} - \bar{\tau}'\mathcal{P} + q\bar{\rho}\bar{\tau}'\}\eta \tag{36}$$

$$[\mathcal{D}, \mathcal{D}']\eta = \{(\bar{\rho}' - \rho')\mathcal{P} + (\rho - \bar{\rho}')\mathcal{P}' - p(\rho\rho' + \Psi_2) - q(\bar{\rho}\bar{\rho}' + \Psi_2)\}\eta \tag{37}$$

together with the remaining commutator relations obtained by applying prime, complex conjugation, and both to equation (36).

*GHP Vacuum Bianchi Identities*

$$\mathcal{D}\Psi_4 = \tau\Psi_4 + 3\kappa'\Psi_2 \tag{38}$$

$$\mathcal{P}\Psi_2 = 3\rho\Psi_2 \tag{39}$$

$$\mathcal{P}'\Psi_2 = 3\rho'\Psi_2 \tag{40}$$

$$\mathcal{P}\Psi_4 = 3\sigma'\Psi_2 + \rho\Psi_4 \tag{41}$$

$$\mathcal{D}'\Psi_2 = 3\tau'\Psi_2 \tag{42}$$

$$\mathcal{D}\Psi_2 = 3\tau\Psi_2 \tag{43}$$

**(c) Petrov Type D :** Here

$$\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0 \; , \; \Psi_2 \neq 0$$
$$\kappa = \sigma = \kappa' = \sigma' = 0.$$

*GHP Field Equations*

$$\mathcal{P}\rho = \rho^2 \tag{44}$$

$$\mathcal{P}'\rho' = \rho'^2 \tag{45}$$

$$\mathcal{P}\tau = (\tau - \bar{\tau}')\rho \tag{46}$$

$$\mathcal{P}'\tau' = (\tau' - \bar{\tau})\rho' \tag{47}$$

$$\mathcal{D}\rho = (\rho - \bar{\rho})\tau \tag{48}$$

$$\mathcal{D}'\rho' = (\rho' - \bar{\rho}')\tau' \tag{49}$$

$$\mathcal{D}\tau = \tau^2 \tag{50}$$

$$\mathcal{D}'\tau' = \tau'^2 \tag{51}$$

$$\mathcal{P}'\rho - \mathcal{D}'\tau = -\rho\bar{\rho}' - \tau\bar{\tau} - \Psi_2 \tag{52}$$

$$\mathcal{P}\rho' - \mathcal{D}\tau' = -\rho'\bar{\rho} - \tau'\bar{\tau}' - \Psi_2 \tag{53}$$

*GHP Commutator Relations*

$$[\mathcal{P}, \mathcal{P}']\eta = \{(\bar{\tau} - \tau')\mathcal{D} + (\tau - \bar{\tau}')\mathcal{D}' - p(-\tau\tau' + \Psi_2) - q(-\bar{\tau}\bar{\tau}' + \Psi_2)\}\eta \tag{54}$$

$$[\mathcal{P}, \mathcal{D}]\eta = \{\bar{\rho}\mathcal{D} - \bar{\tau}'\mathcal{P} - q(-\bar{\rho}\bar{\tau}')\}\eta \tag{55}$$

$$[\mathcal{D}, \mathcal{D}']\eta = \{(\bar{\rho}' - \rho')\mathcal{P} + (\rho - \bar{\rho}')\mathcal{P}' - p(\rho\rho' + \Psi_2) - q(\bar{\rho}\bar{\rho}' + \Psi_2)\}\eta \tag{56}$$

together with the remaining commutator relations obtained by applying prime, complex conjugation, and both to equation (55).

*GHP Vacuum Bianchi Identities*

$$\mathcal{P}\Psi_2 = 3\rho\Psi_2 \tag{57}$$

$$\mathcal{P}'\Psi_2 = 3\rho'\Psi_2 \tag{58}$$

$$\mathcal{D}'\Psi_2 = 3\tau'\Psi_2 \tag{59}$$

$$\mathcal{D}\Psi_2 = 3\tau\Psi_2 \tag{60}$$

**(d) Petrov Type III :** Here

$$\Psi_0 = \Psi_1 = \Psi_2 = \Psi_4 = 0\ ,\ \Psi_3 \neq 0$$

$$\kappa = \sigma = 0.$$

*GHP Field Equations*

$$\mathcal{P}\rho = \rho^2 \tag{61}$$

$$\mathcal{P}'\rho' - \mathcal{D}'\kappa' = \rho'^2 + \sigma'\bar{\sigma}' - \bar{\kappa}'\tau' - \tau\kappa' \tag{62}$$

$$\mathcal{P}'\sigma' - \mathcal{D}'\kappa' = (\rho' + \bar{\rho}')\sigma' - (\tau' + \bar{\tau})\kappa' \tag{63}$$

$$\mathcal{P}\tau = (\tau - \bar{\tau}')\rho \tag{64}$$

$$\mathcal{P}'\tau' - \mathcal{P}\kappa' = (\tau' - \bar{\tau})\rho' + (\bar{\tau}' - \tau)\sigma' + \Psi_3 \tag{65}$$

$$\mathcal{D}\rho = (\rho - \bar{\rho})\tau \tag{66}$$

$$\mathcal{D}'\rho' - \mathcal{D}\sigma' = (\rho' - \bar{\rho}')\tau' + (\rho - \bar{\rho})\kappa' + \Psi_3 \tag{67}$$

$$\mathcal{D}\tau = -\bar{\sigma}'\rho + \tau^2 \tag{68}$$

$$\mathcal{D}'\tau' - \mathcal{P}\sigma' = -\rho\sigma' + \tau'^2 \tag{69}$$

$$\mathcal{P}'\rho - \mathcal{D}'\tau = -\rho\bar{\rho}' - \tau\bar{\tau} \tag{70}$$

$$\mathcal{P}\rho' - \mathcal{D}\tau' = -\rho'\bar{\rho} - \tau'\bar{\tau}' \tag{71}$$

*GHP Commutator Relations*

$$[\mathcal{P}, \mathcal{P}']\eta = \{(\bar{\tau} - \tau')\mathcal{D} + (\tau - \bar{\tau}')\mathcal{D}' - p(-\tau\tau') - q(-\bar{\tau}\bar{\tau}')\}\eta \tag{72}$$

$$[\mathcal{P}, \mathcal{D}]\eta = \{\bar{\rho}\mathcal{D} - \bar{\tau}'\mathcal{P} - q(-\bar{\rho}\bar{\tau}')\}\eta \tag{73}$$

$$[\mathcal{D}, \mathcal{D}']\eta = \{(\bar{\rho}' - \rho')\mathcal{P} + (\rho - \bar{\rho}')\mathcal{P}' - p(\rho\rho') - q(\bar{\rho}\bar{\rho}')\}\eta \tag{74}$$

together with the remaining commutator relations obtained by applying prime, complex conjugation, and both to equation (73).

*GHP Vacuum Bianchi Identities*

$$\mathcal{P}'\Psi_3 = 4\rho'\Psi_3 \tag{75}$$

$$\mathcal{D}\Psi_3 = 2\tau'\Psi_3 \tag{76}$$

$$\mathcal{D}'\Psi_3 = 4\tau'\Psi_3 \tag{77}$$

$$\mathcal{P}\Psi_3 = 2\rho\Psi_3 \tag{78}$$

**(e) Petrov Type N :** Here

$$\Psi_0 \ = \ \Psi_1 \ = \ \Psi_2 \ = \ \Psi_3 \ = \ 0 \ , \ \Psi_4 \neq 0$$

$$\kappa \ = \ \sigma \ = \ 0.$$

*GHP Field Equations*

$$\mathcal{P}\rho = \rho^2 \tag{79}$$

$$\mathcal{P}'\rho' - \mathcal{D}'\kappa' = \rho'^2 + \sigma'\bar{\sigma}' - \bar{\kappa}'\tau' - \tau\kappa' \tag{80}$$

$$\mathcal{P}'\sigma' - \mathcal{D}'\kappa' = (\rho' + \bar{\rho}')\sigma' - (\tau' + \bar{\tau})\kappa' + \Psi_4 \tag{81}$$

$$\mathcal{P}\tau = (\tau - \bar{\tau}')\rho \tag{82}$$

$$\mathcal{P}'\tau' - \mathcal{P}\kappa' = (\tau' - \bar{\tau})\rho' + (\bar{\tau}' + \tau)\sigma' \tag{83}$$

$$\mathcal{D}\rho = (\rho - \bar{\rho})\tau \tag{84}$$

$$\mathcal{D}'\rho' - \mathcal{D}\sigma' = (\rho' - \bar{\rho}')\tau' + (\rho - \bar{\rho})\kappa' \tag{85}$$

$$\mathcal{D}\tau = -\bar{\sigma}'\rho + \tau^2 \tag{86}$$

$$\mathcal{D}'\tau' - \mathcal{P}\sigma' = -\rho\sigma' + \tau'^2 \tag{87}$$

$$\mathcal{P}'\rho - \mathcal{D}'\tau = -\rho\bar{\rho}' - \tau\bar{\tau} \tag{88}$$

$$\mathcal{P}\rho' - \mathcal{D}\tau' = -\rho'\bar{\rho} - \tau'\bar{\tau}' \tag{89}$$

*GHP Commutator Relations*

$$[\mathcal{P},\mathcal{P}']\eta = \{(\bar{\tau} - \tau')\mathcal{D} + (\tau - \bar{\tau}')\mathcal{D}' - p(-\tau\tau') - q(-\bar{\tau}\bar{\tau}')\}\eta \tag{90}$$

$$[\mathcal{P},\mathcal{D}]\eta = \{\bar{\rho}\mathcal{D} - \bar{\tau}'\mathcal{P} - q(-\bar{\rho}\bar{\tau}')\}\eta \tag{91}$$

$$[\mathcal{D},\mathcal{D}']\eta = \{(\bar{\rho}' - \rho')\mathcal{P} + (\rho - \bar{\rho}')\mathcal{P}' - p(\rho\rho') - q(\bar{\rho}\bar{\rho}')\}\eta \tag{92}$$

together with the remaining commutator relations obtained by applying prime, complex conjugation, and both to equation (91).

*GHP Vacuum Bianchi Identities*

$$\mathcal{D}\Psi_4 = \tau\Psi_4 \tag{93}$$

$$\mathcal{P}\Psi_4 = \rho\Psi_4 \tag{94}$$

## 2. Weyl Scalars and Spin Coefficients

The Weyl scalars $\Psi_i$ , $(i = 0, 1, .., 4)$, in terms of the spin coefficients and their derivatives can be expressed as follows :

$$\Psi_0 = (\mathcal{P} - \rho - \bar{\rho})\sigma - (\mathcal{D} - \tau - \bar{\tau}')\kappa \tag{95}$$

$$\Psi_1 = (\mathcal{P} - \rho)\tau - \mathcal{P}'\kappa + \bar{\tau}'\rho - (\bar{\tau} - \tau')\sigma \tag{96}$$

$$\Psi_1 = (\mathcal{D} - \tau)\rho - \mathcal{D}'\sigma + \bar{\rho}\tau - (\rho' - \bar{\rho}')\kappa \tag{97}$$

$$\Psi_2 = -(\mathcal{P}' + \bar{\rho}')\rho + (\mathcal{D}' - \bar{\tau})\tau - \kappa\kappa' \tag{98}$$

$$\Psi_2 = -(\mathcal{P} + \bar{\rho})\rho' + (\mathcal{D} - \bar{\tau}')\tau' - \kappa'\kappa \tag{99}$$

$$\Psi_3 = (\mathcal{P}' - \rho')\tau' - \mathcal{P}\kappa' + \bar{\tau}\rho' - (\bar{\tau}' - \tau)\sigma' \tag{100}$$

$$\Psi_3 = (\mathcal{D}' - \tau')\rho' - \mathcal{D}\sigma' + \bar{\rho}'\tau' - (\rho - \bar{\rho})\kappa' \tag{101}$$

$$\Psi_4 = (\mathcal{P}' - \rho' - \bar{\rho}')\sigma' - (\mathcal{D}' - \tau' - \bar{\tau})\kappa' \tag{102}$$

From equations (95) - (102) it may be noted that

$$\begin{aligned} \Psi_0 &= \Psi_4' \\ \Psi_1 &= \Psi_3' \\ \Psi_2 &= \Psi_2' \\ \Psi_3 &= \Psi_1' \\ \Psi_4 &= \Psi_0' \end{aligned} \tag{103}$$

## 3. Tetrad Vectors and Spin Coefficients

From the definition of the covariant differentiation operators

$$D = l^a\nabla_a \ , \ D' = n^a\nabla_a \ , \ \delta = m^a\nabla_a \ , \ \delta' = \bar{m}^a\nabla_a \tag{104}$$

along the direction of the vectors of a complex null tetrad, it is possible to write equations (44 - 45 a - IV) in the following convenient forms

$$Dl^a = (\epsilon + \bar{\epsilon})l^a - \bar{\kappa}m^a - \kappa\bar{m}^a \tag{105 a}$$

$$D'l^a = (\gamma + \bar{\gamma})l^a - \bar{\tau}m^a - \tau\bar{m}^a \tag{105 b}$$

$$\delta l^a = (\bar{\alpha} + \beta)l^a - \bar{\rho}m^a - \sigma\bar{m}^a \tag{105 c}$$

$$\delta' l^a = (\alpha + \bar{\beta})l^a - \bar{\sigma}m^a - \rho\bar{m}^a \tag{105 d}$$

$$Dn^a = -(\epsilon + \bar{\epsilon})n^a + \pi m^a + \bar{\pi}\bar{m}^a \tag{106 a}$$

$$D'n^a = -(\gamma + \bar{\gamma})n^a + \nu m^a + \bar{\nu}\bar{m}^a \tag{106 b}$$

$$\delta n^a = -(\bar{\alpha} + \beta)n^a + \mu m^a + \bar{\lambda}\bar{m}^a \tag{106 c}$$

$$\delta' n^a = -(\alpha + \bar{\beta})l^a + \lambda m^a + \bar{\mu}\bar{m}^a \tag{106 d}$$

$$Dm^a = \bar{\pi}l^a - \kappa n^a + (\epsilon - \bar{\epsilon})m^a \tag{107 a}$$

$$D'm^a = \bar{\nu}l^a - \tau n^a + (\gamma - \bar{\gamma})m^a \tag{107 b}$$

$$\delta m^a = \bar{\lambda}l^a - \sigma n^a + (\beta - \bar{\alpha})m^a \tag{107 c}$$

$$\delta' m^a = \bar{\mu}l^a - \rho n^a + (\alpha - \bar{\beta})m^a \tag{107 d}$$

$$D\bar{m}^a = \pi l^a - \bar{\kappa} n^a + (\bar{\epsilon} - \epsilon)\bar{m}^a \qquad (108\ a)$$

$$D'\bar{m}^a = \nu l^a - \bar{\tau} n^a + (\bar{\gamma} - \gamma)\bar{m}^a \qquad (108\ b)$$

$$\delta\bar{m}^a = \mu l^a - \bar{\rho} n^a + (\bar{\alpha} - \beta)\bar{m}^a \qquad (108\ c)$$

$$\delta'\bar{m}^a = \lambda l^a - \bar{\sigma} n^a + (\bar{\beta} - \alpha)\bar{m}^a \qquad (108\ d)$$

Moreover, the covariant derivative of an arbitrary scalar function $\eta$ can be written in the form $\nabla_a \eta = C_\mu Z^\mu_a$, where $C_\mu$ are four coefficients to be determined. But from the definition of the anholonomic derivatives $D, D', \delta$ and $\delta'$ the above equation reduces to

$$\nabla_a = l_a D' + n_a D - \bar{m}_a \delta - m_a \delta' \qquad (109)$$

## 4. Lanczos Differential Gauge Conditions, Weyl-Lanczos Relations and Petrov Classifiaction

In this section, we shall find the forms of the GHP versions of the Weyl-Lanczos relations (79 - IV) and Lanczos differential gauge conditions (81 - IV) for the different types of the Petrov classification.

### (a) Petrov Type I

*Weyl-Lanczos Relations*

$$2\mathcal{D}L_0 - 2\mathcal{P}L_4 - 2\bar{\tau}'L_0 + 6\sigma L_1 + 2\bar{\rho}L_4 + 6\kappa L_5 - \Psi_0 = 0 \qquad (110\ a)$$

$$\mathcal{P}'L_0 + \mathcal{D}L_1 - \mathcal{D}'L_4 - \mathcal{P}L_5 + (\rho' - \bar{\rho}')L_0 + (3\tau - \bar{\tau}')L_1$$
$$+2\sigma L_2 + (\bar{\tau} - \tau')L_4 + (\bar{\rho} - 3\rho)L_5 - 2\kappa L_6 = 0 \qquad (110\ b)$$

$$-\mathcal{P}'L_1 + \mathcal{D}L_2 - \mathcal{D}'L_5 - \mathcal{P}L_6 + \kappa' L_0 + (2\rho' - \bar{\rho}')L_1 + (2\tau - \bar{\tau}')L_2$$
$$+\sigma L_3 - \sigma' L_4 + (\bar{\tau} - 2\tau')L_5 + (\bar{\rho} - 2\rho)L_6 - \kappa L_7 - \Psi_2 = 0 \qquad (110\ c)$$

$$\mathcal{P}'L_2 + \mathcal{D}L_3 - \mathcal{D}'L_6 - \mathcal{P}L_7 + 2\kappa' L_1 + (3\rho' - \bar{\rho}')L_2 + (\tau - \bar{\tau}')L_3$$
$$-2\sigma' L_5 + (\bar{\tau} - 3\tau')L_6 + (\bar{\rho} - \rho)L_7 = 0 \qquad (110\ d)$$

$$2\mathcal{P}'L_3 - 2\mathcal{D}'L_7 + 6\kappa' L_2 - 2\bar{\rho}'L_3 - 6\sigma' L_6 + 2\bar{\tau}L_7 - \Psi_4 = 0 \qquad (110\ e)$$

*Lanczos Differential Gauge Conditions*

$$\mathcal{P}'L_2 + \mathcal{D}L_3 - \mathcal{D}'L_6 - \mathcal{P}L_7 + 2\kappa' L_1 - (\rho' + \bar{\rho}')L_2 + (\tau - \bar{\tau}')L_3$$
$$-2\sigma' L_5 + (\bar{\tau} - 3\tau')L_6 - (\bar{\rho} + \rho)L_7 = 0 \qquad (111\ a)$$

$$\mathcal{P}'L_0 + \mathcal{D}L_1 - \mathcal{D}'L_4 + \mathcal{P}L_5 - (\rho' + \bar{\rho}')L_0 + (3\tau + \bar{\tau}')L_1$$
$$-2\sigma L_2 + (\bar{\tau} + \tau')L_4 - (\bar{\rho} + 3\rho)L_5 + 2\kappa L_6 = 0 \qquad (111\ b)$$

$$\mathcal{P}'L_1 - \mathcal{D}L_2 - \mathcal{D}'L_5 - \mathcal{P}L_6 + \kappa' L_0 - (2\rho' + \bar{\rho}')L_1 + (2\tau - \bar{\tau}')L_2$$
$$-\sigma L_3 - \sigma' L_4 + (\bar{\tau} + 2\tau')L_5 - (\bar{\rho} + 2\rho)L_6 + \kappa L_7 = 0 \qquad (111\ c)$$

### (b) **Petrov Type II**

*Weyl-Lanczos Relations*

$$2\mathcal{D}L_0 - 2\mathcal{P}L_4 - 2\bar{\tau}'L_0 + 2\bar{\rho}L_4 = 0 \qquad (112\ a)$$

$$\begin{aligned}\mathcal{P}'L_0 + \mathcal{D}L_1 - \mathcal{D}'L_4 - \mathcal{P}L_5 + (\rho' - \bar{\rho}')L_0 + (3\tau - \bar{\tau}')L_1 \\ +(\bar{\tau} - \tau')L_4 + (\bar{\rho} - 3\rho)L_5\end{aligned} \qquad (112\ b)$$

$$\begin{aligned}-\mathcal{P}'L_1 + \mathcal{D}L_2 - \mathcal{D}'L_5 - \mathcal{P}L_6 + \kappa'L_0 + (2\rho' - \bar{\rho}')L_1 + (2\tau - \bar{\tau}')L_2 \\ -\sigma'L_4 + (\bar{\tau} - 2\tau')L_5 + (\bar{\rho} - 2\rho)L_6 - \Psi_2 = 0\end{aligned} \qquad (112\ c)$$

$$\begin{aligned}\mathcal{P}'L_2 + \mathcal{D}L_3 - \mathcal{D}'L_6 - \mathcal{P}L_7 + 2\kappa'L_1 + (3\rho' - \bar{\rho}')L_2 + (\tau - \bar{\tau}')L_3 \\ -2\sigma'L_5 + (\bar{\tau} - 3\tau')L_6 + (\bar{\rho} - \rho)L_7 = 0\end{aligned} \qquad (112\ d)$$

$$2\mathcal{P}'L_3 - 2\mathcal{D}'L_7 + 6\kappa'L_2 - 2\bar{\rho}'L_3 - 6\sigma'L_6 + 2\bar{\tau}L_7 - \Psi_4 = 0 \qquad (112\ e)$$

*Lanczos Differential Gauge Conditions*

$$\begin{aligned}\mathcal{P}'L_2 + \mathcal{D}L_3 - \mathcal{D}'L_6 - \mathcal{P}L_7 + 2\kappa'L_1 - (\rho' + \bar{\rho}')L_2 + (\tau - \bar{\tau}')L_3 \\ -2\sigma'L_5 + (\bar{\tau} - 3\tau')L_6 - (\bar{\rho} + \rho)L_7 = 0\end{aligned} \qquad (113\ a)$$

$$\begin{aligned}\mathcal{P}'L_0 + \mathcal{D}L_1 - \mathcal{D}'L_4 + \mathcal{P}L_5 - (\rho' + \bar{\rho}')L_0 + (3\tau + \bar{\tau}')L_1 \\ +(\bar{\tau} + \tau')L_4 - (\bar{\rho} + 3\rho)L_5 = 0\end{aligned} \qquad (113\ b)$$

$$\begin{aligned}\mathcal{P}'L_1 - \mathcal{D}L_2 - \mathcal{D}'L_5 - \mathcal{P}L_6 + \kappa'L_0 - (2\rho' + \bar{\rho}')L_1 + (2\tau - \bar{\tau}')L_2 \\ -\sigma'L_4 + (\bar{\tau} + 2\tau')L_5 - (\bar{\rho} + 2\rho)L_6 = 0\end{aligned} \qquad (113\ c)$$

### (c) Petrov Type D

*Weyl-Lanczos Relations*

$$2\mathcal{D}L_0 - 2\mathcal{P}L_4 - 2\bar{\tau}'L_0 + 2\bar{\rho}L_4 = 0 \qquad (114\ a)$$

$$\begin{aligned}\mathcal{P}'L_0 + \mathcal{D}L_1 - \mathcal{D}'L_4 - \mathcal{P}L_5 + (\rho' - \bar{\rho}')L_0 + (3\tau - \bar{\tau}')L_1 \\ +(\bar{\tau} - \tau')L_4 + (\bar{\rho} - 3\rho)L_5 = 0\end{aligned} \qquad (114\ b)$$

$$\begin{aligned}-\mathcal{P}'L_1 + \mathcal{D}L_2 - \mathcal{D}'L_5 - \mathcal{P}L_6 + (2\rho' - \bar{\rho}')L_1 + (2\tau - \bar{\tau}')L_2 \\ +(\bar{\tau} - 2\tau')L_5 + (\bar{\rho} - 2\rho)L_6 - \Psi_2 = 0\end{aligned} \qquad (114\ c)$$

$$\begin{aligned}\mathcal{P}'L_2 + \mathcal{D}L_3 - \mathcal{D}'L_6 - \mathcal{P}L_7 + (3\rho' - \bar{\rho}')L_2 + (\tau - \bar{\tau}')L_3 \\ +(\bar{\tau} - 3\tau')L_6 + (\bar{\rho} - \rho)L_7 = 0\end{aligned} \qquad (114\ d)$$

$$2\mathcal{P}'L_3 - 2\mathcal{D}'L_7 - 2\bar{\rho}'L_3 + 2\bar{\tau}L_7 = 0 \qquad (114\ e)$$

*Lanczos Differential Gauge Conditions*

$$\begin{aligned}\mathcal{P}'L_2 + \mathcal{D}L_3 - \mathcal{D}'L_6 - \mathcal{P}L_7 - (\rho' + \bar{\rho}')L_2 + (\tau - \bar{\tau}')L_3 \\ +(\bar{\tau} - 3\tau')L_6 - (\bar{\rho} + \rho)L_7 = 0\end{aligned} \qquad (115\ a)$$

$$\begin{aligned}\mathcal{P}'L_0 + \mathcal{D}L_1 - \mathcal{D}'L_4 + \mathcal{P}L_5 - (\rho' + \bar{\rho}')L_0 + (3\tau + \bar{\tau}')L_1 \\ +(\bar{\tau} + \tau')L_4 - (\bar{\rho} + 3\rho)L_5 = 0\end{aligned} \qquad (115\ b)$$

$$\mathcal{P}'L_1 - \mathcal{D}L_2 - \mathcal{D}'L_5 - \mathcal{P}L_6 - (2\rho' + \bar{\rho}')L_1 + (2\tau - \bar{\tau}')L_2$$
$$+(\bar{\tau} + 2\tau')L_5 - (\bar{\rho} + 2\rho)L_6 = 0 \qquad (115\ c)$$

**Petrov Type III**

*Weyl-Lanczos Relations*

$$2\mathcal{D}L_0 - 2\mathcal{P}L_4 - 2\bar{\tau}'L_0 + 2\bar{\rho}L_4 = 0 \qquad (116\ a)$$

$$\mathcal{P}'L_0 + \mathcal{D}L_1 - \mathcal{D}'L_4 - \mathcal{P}L_5 + (\rho' - \bar{\rho}')L_0 + (3\tau - \bar{\tau}')L_1$$
$$+(\bar{\tau} - \tau')L_4 + (\bar{\rho} - 3\rho)L_5 = 0 \qquad (116\ b)$$

$$-\mathcal{P}'L_1 + \mathcal{D}L_2 - \mathcal{D}'L_5 - \mathcal{P}L_6 + \kappa'L_0 + (2\rho' - \bar{\rho}')L_1 + (2\tau - \bar{\tau}')L_2$$
$$-\sigma'L_4 + (\bar{\tau} - 2\tau')L_5 + (\bar{\rho} - 2\rho)L_6 = 0 \qquad (116\ c)$$

$$\mathcal{P}'L_2 + \mathcal{D}L_3 - \mathcal{D}'L_6 - \mathcal{P}L_7 + 2\kappa'L_1 + (3\rho' - \bar{\rho}')L_2 + (\tau - \bar{\tau}')L_3$$
$$-2\sigma'L_5 + (\bar{\tau} - 3\tau')L_6 + (\bar{\rho} - \rho)L_7 - \Psi_3 = 0 \qquad (116\ d)$$

$$2\mathcal{P}'L_3 - 2\mathcal{D}'L_7 + 6\kappa'L_2 - 2\bar{\rho}'L_3 - 6\sigma'L_6 + 2\bar{\tau}L_7 = 0 \qquad (116\ e)$$

*Lanczos Differential Gauge Conditions*

$$\mathcal{P}'L_2 + \mathcal{D}L_3 - \mathcal{D}'L_6 - \mathcal{P}L_7 + 2\kappa'L_1 - (\rho' + \bar{\rho}')L_2 + (\tau - \bar{\tau}')L_3$$
$$-2\sigma'L_5 + (\bar{\tau} - 3\tau')L_6 - (\bar{\rho} + \rho)L_7 = 0 \qquad (117\ a)$$

$$\mathcal{P}'L_0 + \mathcal{D}L_1 - \mathcal{D}'L_4 + \mathcal{P}L_5 - (\rho' + \bar{\rho}')L_0 + (3\tau + \bar{\tau}')L_1$$
$$+(\bar{\tau} + \tau')L_4 - (\bar{\rho} + 3\rho)L_5 = 0 \qquad (117\ b)$$

$$\mathcal{P}'L_1 - \mathcal{D}L_2 - \mathcal{D}'L_5 - \mathcal{P}L_6 + \kappa' L_0 - (2\rho' + \bar{\rho}')L_1 + (2\tau - \bar{\tau}')L_2$$
$$-\sigma' L_4 + (\bar{\tau} + 2\tau')L_5 - (\bar{\rho} + 2\rho)L_6 = 0 \qquad (117\ c)$$

**Petrov Type N**

*Weyl-Lanczos Relations*

$$2\mathcal{D}L_0 - 2\mathcal{P}L_4 - 2\bar{\tau}'L_0 + 2\bar{\rho}L_4 = 0 \qquad (118\ a)$$

$$\mathcal{P}'L_0 + \mathcal{D}L_1 - \mathcal{D}'L_4 - \mathcal{P}L_5 + (\rho' - \bar{\rho}')L_0 + (3\tau - \bar{\tau}')L_1$$
$$+(\bar{\tau} - \tau')L_4 + (\bar{\rho} - 3\rho)L_5 = 0 \qquad (118\ b)$$

$$-\mathcal{P}'L_1 + \mathcal{D}L_2 - \mathcal{D}'L_5 - \mathcal{P}L_6 + \kappa' L_0 + (2\rho' - \bar{\rho}')L_1 + (2\tau - \bar{\tau}')L_2$$
$$-\sigma' L_4 + (\bar{\tau} - 2\tau')L_5 + (\bar{\rho} - 2\rho)L_6 = 0 \qquad (118\ c)$$

$$\mathcal{P}'L_2 + \mathcal{D}L_3 - \mathcal{D}'L_6 - \mathcal{P}L_7 + 2\kappa' L_1 + (3\rho' - \bar{\rho}')L_2 + (\tau - \bar{\tau}')L_3$$
$$-2\sigma' L_5 + (\bar{\tau} - 3\tau')L_6 + (\bar{\rho} - \rho)L_7 = 0 \qquad (118\ d)$$

$$2\mathcal{P}'L_3 - 2\mathcal{D}'L_7 + 6\kappa' L_2 - 2\bar{\rho}'L_3 - 6\sigma' L_6 + 2\bar{\tau}L_7 - \Psi_4 = 0 \qquad (118\ e)$$

*Lanczos Differential Gauge Conditions*

$$\mathcal{P}'L_2 + \mathcal{D}L_3 - \mathcal{D}'L_6 - \mathcal{P}L_7 + 2\kappa' L_1 - (\rho' + \bar{\rho}')L_2 + (\tau - \bar{\tau}')L_3$$
$$-2\sigma' L_5 + (\bar{\tau} - 3\tau')L_6 - (\bar{\rho} + \rho)L_7 = 0 \qquad (119\ a)$$

$$\mathcal{P}'L_0 + \mathcal{D}L_1 - \mathcal{D}'L_4 + \mathcal{P}L_5 - (\rho' + \bar{\rho}')L_0 + (3\tau + \bar{\tau}')L_1$$
$$+(\bar{\tau} + \tau')L_4 - (\bar{\rho} + 3\rho)L_5 = 0 \qquad (119\ b)$$

$$\mathcal{P}'L_1 - \mathcal{D}L_2 - \mathcal{D}'L_5 - \mathcal{P}L_6 + \kappa'L_0 - (2\rho' + \bar{\rho}')L_1 + (2\tau - \bar{\tau}')L_2$$
$$-\sigma'L_4 + (\bar{\tau} + 2\tau')L_5 - (\bar{\rho} + 2\rho)L_6 = 0 \qquad (119\ c)$$

## 5. Analogies Between Electromagnetism and Gravitation

The 'Maxwell-like' equation (22 - IV) and the Weyl-Lanczos relations (16 - IV) suggest that there is a close analogy between electromagnetic radiation via the 4-potential and the gravitational radiation via the Lanczos potential.

The field equations for an electromagnetic field $F_{ij}$ ( $= -F_{ij}$ ) are

$$F_{ij;k} \;+\; F_{jk;i} \;+\; F_{ki;j} \;=\; 0 \qquad (120)$$

$$F^{ik}{}_{;k} \;=\; J^i \qquad (121)$$

Since the equations (120) are equivalent to the local existence of a 4- potential $A_i$ so that

$$F_{ij} \;=\; W_{ij}(A) \;=\; A_{i;j} \;-\; A_{j;i} \qquad (122)$$

we can omit equation (120) and replace it by equation (122). Thus equations (121) and (122) generalize to the field equations (22 - IV) and the Weyl-Lanczos relations (16 - IV), respectively.

The gravitational counterpart of the electromagnetic gauge invariance

$$A_i \longrightarrow A_i' \;=\; A_i \;+\; \chi_{,i} \;\;,\;\; W_{ij}\,(A') \;=\; W_{ij}\,(A)$$

is given by

$$L_{ijk} \longrightarrow L'_{ijk} = L_{ijk} + \chi_{ijk} \quad , \quad W_{hijk}\,(L') \;=\; W_{hijk}\,(L)$$

while the electromagnetic gauge condition

$$A^{i}{}_{;i} \;=\; 0$$

has the gravitational counterpart as

$$L_{ij}{}^{t}{}_{;t} \;=\; 0$$

The analogies between electromagnetic and gravitational theories have been mentioned in a table on the next page (see also [14]).

**Table**

*A comparison between electromagnetic and gravitational theories*

| | | |
|---|---|---|
| Fields | $F_{ij}$ | $C_{hijk}$ |
| Potentials | $A_i$ | $L_{ijk}$ |
| Field relations | $F_{ij} = A_{i;j} - A_{j;i}$ | $C_{hijk} = W(L)_{hijk}$ |
| Gauge invariance | $A'_i = A_i + \chi_{,i}$ <br> $W_{ij}(A') = W_{ij}(A)$ | $L'_{ijk} = L_{ijk} + \chi_{ijk}$ <br> $W_{hijk}(L') = W_{hijk}(L)$ |
| Gauge conditions | $A^i{}_{;i} = 0$ | $L_{ij}{}^t{}_{;t} = 0$ |
| Field equations | $F^{ij}{}_{;j} = J^i$ | $C_{ijk}{}^t{}_{;t} = J_{ijk}$ |
| Potential wave equation in *matter* | $\Box A_i + R_i{}^k A_k = J_i$ | $\Box L_{ijk} + 2R_k{}^t L_{ijt} - R_i{}^t L_{jkt}$ <br> $-R_j{}^t L_{kit} - g_{ik} R^{pt} L_{pjt}$ <br> $+g_{jk} R^{pt} L_{pit} - \frac{1}{2} R L_{ijk} = J_{ijk}$ |
| Potential wave equation in *vacuo* | $\Box A_i + R_i{}^k A_k = 0$ | $\Box L_{ijk} = 0$ |
| Field wave equation in *matter* | $\Box F_{ij} + R^t{}_i F_{tj} - R^t{}_j F_{ti}$ <br> $-2R_{risj} F^{rs} - F_i{}^t{}_{;t;j}$ <br> $+F_j{}^t{}_{;t;i} = 0$ | $\Box R_{hijk} + 4R_{hpq[j} R_{k]}{}^q{}_i{}^p$ <br> $-R_{hipq} R^{pq}{}_{jk} + 2R^q{}_{[k} R_{j]qhi}$ <br> $+2R_{j[i;h];k} + 2R_{k[h;i];j} = 0$ |
| Field wave equation in *vacuo* | $\Box F_{ij} + R^t{}_i F_{tj} - R^t{}_j F_{ti}$ <br> $-2R_{risj} F^{rs} = 0$ | $\Box C_{hijk} + 4C_{hpq[j} C_{k]}{}^q{}_i{}^p$ <br> $-C_{hi}{}^{pq} C_{pqjk} = 0$ |

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